The role of “positivity” in moment and polynomial optimization problems

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Truncated K-Moment Problem

Given an \( n \)-dimensional multisequence of degree \( m \),

\[
\beta \equiv \beta^{(m)} = \{ \beta_i : i \in \mathbb{Z}^n_+, |i| \leq m \},
\]

and a closed set \( K \subseteq \mathbb{R}^n \), find conditions on \( \beta \) so that there exists a positive Borel measure \( \mu \) on \( \mathbb{R}^n \) such that \( \text{supp} \ \mu \subseteq K \) and

\[
\beta_i = \int_{\mathbb{R}^n} x^i \, d\mu(x) \ (|i| \leq m)
\]

\((x \equiv (x_1, \ldots, x_n), i \equiv (i_1, \ldots, i_n) \in \mathbb{Z}^n_+, x^i := x_1^{i_1} \cdots x_n^{i_n}).\)

By analogy, in the Full K-Moment Problem, we are given \( \beta^{(\infty)} \), with moment data of all degrees.
Multisequence notation and moment matrices

Let \( \beta \) denote an \( n \)-dimensional real multisequence of degree \( m \),

\[
\beta \equiv \beta^{(m)} = \{ \beta_i : i \in \mathbb{Z}_+^n, |i| \leq m \},
\]

Example. For \( n = 1, m = 4 \), \( \beta^{(4)} : \beta_0, \ldots, \beta_4 \), we associate \( \beta^{(4)} \) to the moment matrix \( M_2 \), with rows and columns indexed by \( 1, x, x^2 \), defined by

\[
M_2(\beta) \equiv \begin{pmatrix}
\beta_0 & \beta_1 & \beta_2 \\
\beta_1 & \beta_2 & \beta_3 \\
\beta_2 & \beta_3 & \beta_4
\end{pmatrix}.
\]
Example. For $n = 2$, $m = 4$, consider $\beta^{(4)}$:

$\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20}, \beta_{11}, \beta_{02}, \beta_{30}, \beta_{21}, \beta_{12}, \beta_{03}, \beta_{40}, \beta_{31}, \beta_{22}, \beta_{13}, \beta_{04}$.

We associate $\beta^{(4)}$ to the moment matrix $M_2$, with rows and columns indexed by $1, x, y, x^2, xy, y^2$, defined by

$$M_2(\beta) \equiv \begin{pmatrix} 
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} 
\end{pmatrix}.$$
The Polynomial Optimization Problem

Let $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_n]$. For $q_0, \ldots, q_k \in \mathcal{P}$, $q_0 \equiv 1$, let $K_Q$ denote the basic closed semialgebraic set

$$K_Q = \{ x \in \mathbb{R}^n : q_i(x) \geq 0 \ (0 \leq i \leq k) \}.$$

For $p \in \mathcal{P}$, we seek to compute (or estimate)

$$p_* := \inf_{x \in K_Q} p(x).$$

Later, we will discuss an algorithm of J.-B. Lasserre [2000] which estimates $p_*$ based on “moment relaxations”, and whose stopping criterion (when $p_*$ is computed exactly) is based on the theory of the truncated $K$-moment problem.
Representing measures
Suppose we have a measure $\mu$ as above:

$$\beta_i = \int_{\mathbb{R}^n} x^i d\mu(x) \ (|i| \leq m), \ supp \ \mu \subseteq K$$

$\mu$ is a $K$-representing measure for $\beta$.

For $K = \mathbb{R}^n$, $\mu$ is a representing measure.

$\mu$ is a finitely atomic $K$-representing measure if

$$\mu = \sum_{i=1}^{k} \rho_i \delta_{w_i} \ (\rho_i > 0, w_i \in K).$$
**Question** If there is a $K$-representing measure, is there a finitely atomic $K$-representing measure?

A theorem of V. Tchakaloff [1957] provides an affirmative answer for $K$ compact. The complete answer was found 50 years later:

**Theorem** [C. Bayer and J. Teichmann, 2006]

If $\beta^{(m)}$ has a $K$-representing measure, then $\beta$ has a finitely-atomic $K$-representing measure $\mu$, with $\text{card supp } \mu \leq \text{dim } \mathbb{R}_m[x_1, \ldots, x_n]$
The Full K-Moment Problem

\[ \beta \equiv \beta^{(\infty)} = \{ \beta_i : i \in \mathbb{Z}_+^n \} \]

Stieltjes [1894] \( K = [0, +\infty) \)

Hamburger [1920] \( K = \mathbb{R} \)

Hausdorff [1923] \( K = [a, b] \)

Results for FMP suggest results for TMP.
Connection between TMP and FMP

**Theorem** [Jan Stochel, 2001]

The full multisequence $\beta \equiv \beta^{(\infty)}$ has a $K$-representing measure if and only if $\beta^{(m)}$ has a $K$-representing measure for every $m \geq 1$.

In some cases (one of which is illustrated below), we can use solutions to TMP, together with Stochel’s theorem, to solve FMP.
Riesz functional

\[ \mathcal{P} := \mathbb{R}[x_1, \ldots, x_n] \]

\[ \beta \equiv \beta^{(\infty)} = \{ \beta_i : i \in \mathbb{Z}_+^n \} \]

Riesz functional: \( L_\beta : \mathcal{P} \rightarrow \mathbb{R} \)

\[ p \equiv \sum a_i x^i \rightarrow L_\beta(\sum a_i x^i) = \sum a_i \beta_i \quad (= \int_K p(x) d\mu(x)) \]

Note: If \( \beta \) has a \( K \)-rep. measure \( \mu \), then \( L_\beta \) is \( K \)-positive, i.e.,

\[ p|_K \geq 0 \implies L_\beta(p) \geq 0. \]

(For \( K = \mathbb{R}^n \), we say \( L_\beta \) is positive.)
“Abstract” solution of the Full K-Moment Problem

**Theorem** [M. Riesz, 1923 ($n = 1$), E.K. Haviland, 1936 ($n \geq 2$)]

The full multisequence $\beta \equiv \beta^{(\infty)}$ has a $K$-representing measure if and only if $L_\beta$ is $K$-positive, i.e.,

$$p \in \mathcal{P}, \ p|_K \geq 0 \implies L_\beta(p) \geq 0.$$ 

**A limitation of Riesz-Haviland:**

For a general closed set $K$ (even for $\mathbb{R}^2$), there is no concrete structure theorem for $K$-positive polynomials, so it is difficult to check that $L_\beta$ is $K$-positive.
Moment matrices

Given $\beta \equiv \beta^{(\infty)}$, we define the moment matrix $M \equiv M_\infty(\beta)$:

$$M_\infty(\beta) = (\beta_{i+j})_{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+}.$$

$M_\infty(\beta)$ is uniquely determined by

$$\langle M_\infty(\beta) \hat{s}, \hat{q} \rangle = L_\beta(pq) \quad \forall p, q \in \mathcal{P},$$

where $\hat{s}$ denotes the coefficient vector of $s \in \mathcal{P}$ relative to the basis of monomials in degree-lexicographic order.

If $L_\beta$ is positive (in particular, if $\beta$ has a representing measure), then $M_\infty(\beta) \succeq 0$ (positive semidefinite):

$$\langle M_\infty(\beta) \hat{s}, \hat{s} \rangle = L_\beta(p^2) = \int p^2 d\mu \geq 0.$$

Summary so far: rep. meas. $\iff L_\beta$ pos. $\implies M \succeq 0.$
**Sums of squares**

There is one situation where the “concrete” condition $M \succeq 0$ readily implies that $L_\beta$ is positive. Consider the following property:

$(H_{n,d})$ Every $p \in \mathcal{P}_d$ with $p|\mathbb{R}^n \geq 0$ can be expressed as $p = \sum p_i^2$. If $(H_{n,d})$ holds and $M \succeq 0$, then $L_\beta$ is positive:

$$L_\beta(p) = L_\beta(\sum p_i^2) = \sum \langle M \hat{p}_i, \hat{p}_i \rangle \geq 0.$$

**Hilbert’s theorem on sums of squares** [D. Hilbert, 1888]

$(H_{n,d})$ holds $\iff n = 1$, or $(n, d) = (2, 4)$, or $d = 2$.

The moment problem can be solved concretely in the positive cases of Hilbert’s theorem; we will discuss the first two cases in the sequel.
We consider FMP in the first case of Hilbert’s theorem, $n = 1$.

**Theorem** [Hamburger, 1920]

Let $n = 1$, $K = \mathbb{R}$. The full multisequence $\beta \equiv \beta^{(\infty)}$ has a representing measure if and only if $M_{\infty}(\beta) \succeq 0$.

**Proof.**

For $p \in \mathbb{R}[x]$, $p|_{\mathbb{R}} \geq 0 \implies p = r^2 + s^2$ for some $r, s \in \mathbb{R}[x]$. Then $L_\beta(p) = L_\beta(r^2) + L_\beta(s^2) = \langle M\hat{r}, \hat{r} \rangle + \langle M\hat{s}, \hat{s} \rangle \geq 0$.

Apply Riesz’ Theorem. $\square$
Conditions for solving TMP (with R. Curto)

K-positivity in TMP

\[ \beta \equiv \beta(m) \]

\[ \mathcal{P}_k := \{ p \in \mathcal{P} : \deg p \leq k \} \]

Riesz functional: \( L_\beta : \mathcal{P}_m \to \mathbb{R} \)

\[ p \equiv \sum a_i x^i \mapsto L_\beta(\sum a_i x^i) = \sum a_i \beta_i (= \int_{\mathbb{R}^n} p(x) d\mu(x)) \]

If \( \beta \) has a \( K \)-representing measure, then \( L_\beta \) is \( K \)-positive, i.e.,

\[ p \in \mathcal{P}_m, \quad p|_K \geq 0 \implies L_\beta(p) \geq 0. \]
If $\beta \equiv \beta^{(m)}$ has a $K$-representing measure, then $L_\beta$ is $K$-positive.

Tchakaloff’s Thm. implies that the converse is true for $K$ compact. For the noncompact case, we introduce an example.

For $n = 1$, $K = \mathbb{R}$, $\beta \equiv \beta^{(4)}$, consider the moment problem with

$$\beta_0 = \beta_1 = \beta_2 = \beta_3 = 1, \beta_4 = 2.$$  

We will show below that $L_\beta$ is positive.

Is there a representing measure?

**Question** What is the analogue of R-H for TMP?
Moment matrices for TMP

For $\beta \equiv \beta^{(2d)}$, we define the $d$-th order moment matrix $M_d(\beta)$:

$$M_d(\beta) = (\beta_{i+j})(i,j) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n : |i|, |j| \leq d.$$ 

$M_d(\beta)$ is uniquely determined by

$$\langle M_d(y)\hat{p}, \hat{q} \rangle = L_\beta(pq) \forall p, q \in \mathcal{P}_d.$$
Positivity condition for TMP

Necessary condition 1: positivity
If $\beta$ has a representing measure, then $M_d(\beta) \succeq 0$:

$$\langle M_d(\beta)\hat{\rho}, \hat{\rho}\rangle = L_\beta(p^2) = \int p^2 d\mu \geq 0$$

In the preceding example, we have

$$M_2(\beta) \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

$M(2) \succeq 0$ and therefore (by sos) $L_\beta$ is positive. Does $\beta$ have a representing measure?
Recursiveness

Columns of $M_d(\beta)$: $1, X_1, \ldots, X_n, \ldots, X_1^d, \ldots, X_n^d$

A dependence relation in $\text{Col} M_d(\beta)$ can be denoted by $p(X) \equiv p(X_1, \ldots, X_n) = 0$ for some $p \in \mathcal{P}_d$.

**Necessary condition 2: recursiveness**

$M_d(\beta)$ is recursively generated:

$$p(X) = 0 \implies (pq)(X) = 0 \text{ whenever } pq \in \mathcal{P}_d.$$ 

In the example, $X = 1$, but $X^2 \neq X$, so $L_\beta$ is positive, but there is no measure. The direct analogue of R-H fails for TMP.
An analogue of the Riesz-Haviland theorem for TMP

**Theorem** [C-F, 2008]

Let \( \beta \equiv \beta^{(2d)} \) or \( \beta \equiv \beta^{(2d+1)} \). \( \beta \) has a \( K \)-representing measure \( \iff L_\beta \) admits a \( K \)-positive extension \( L_\beta : \mathcal{P}_{2d+2} \to \mathbb{R} \).

**Issue**: In general it is very difficult to establish \( K \)-positivity.
Example of a concrete truncated moment theorem

We return to the first case of Hilbert’s theorem, \( n = 1 \).

**Theorem** [C-F, 1991]

Let \( n = 1 \). \( \beta \equiv \beta^{(2d)} \) has a representing measure \( \iff M_d(\beta) \succeq 0 \) and \( M_d(\beta) \) is recursively generated.
Using TMP to solve FMP

Another proof of Hamburger’s Theorem

Theorem

Let \( n = 1 \). The full multisequence \( \beta \equiv \beta^{(\infty)} \) has a representing measure if and only if \( M_{\infty}(\beta) \succeq 0 \).

Proof.

Suppose \( M_{\infty} \succeq 0 \). Then, for each \( d \), \( M_{d}(\beta) \) is positive semidefinite and recursively generated, so \( \beta^{(2d)} \) has a representing measure. Stochel’s theorem now implies that \( \beta^{(\infty)} \) has a representing measure. \( \square \)
The variety of a moment matrix

The *variety* of $\beta \equiv \beta^{(2d)}$ (or of $M_d(\beta)$):

$$V(\beta) = \bigcap_{p \in P_d, p(X) = 0} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) = \{x \in \mathbb{R}^n : p(x) = 0\}$. 
The variety condition

Proposition
If \( \mu \) is a representing measure, then \( \text{supp} \ \mu \subseteq \mathcal{V}(\beta) \) and

\[
\text{rank} \ M_d(\beta) \leq \text{card} \ \text{supp} \ \mu \leq \text{card} \ \mathcal{V}(\beta).
\]

Necessary condition 3: variety condition

\[
r \equiv \text{rank} \ M_d(\beta) \leq v \equiv \text{card} \ \mathcal{V}(\beta).
\]

In the previous example, \( r = 2, \ X = 1, \ \mathcal{V}(\beta) = \{1\}, \ v = 1. \)
The flat extension theorem

Recall: If \( \mu \) is a representing measure, then

\[
    r \equiv \text{rank } M_d(\beta) \leq \text{card supp } \mu.
\]

**Theorem [C-F, 1996, 2005]**

\( \beta \equiv \beta^{(2n)} \) has an \( r \)-atomic representing measure \( \iff M_d(\beta) \succeq 0 \) and \( M_d(\beta) \) has a flat, i.e., rank-preserving, moment matrix extension \( M_{d+1} \). In this case, an \( r \)-atomic representing measure \( \mu \) can be explicitly constructed with \( \text{supp } \mu = \mathcal{V}(M_{d+1}) \).
Solution to TMP based on moment matrix extensions

**Theorem [C-F, 2005]**

\[ \beta \equiv \beta^{(2d)} \text{ has a representing measure } \iff M_d(\beta) \text{ admits a positive extension } M_{d+k} \text{ (for some } k \geq 0), \text{ and } M_{d+k} \text{ has a flat extension } M_{d+k+1}. \]

Note: When the strategy of this theorem can be implemented, this method circumvents the difficulty of positivity for \( L_\beta^{(2d+2)} \) in the truncated R-H theorem. In this case, is there some way to recognize directly that for \( M_{d+1} \) (as above), \( L_\beta^{(2d+2)} \) is positive?
TMP for \( K \) a planar curve of degree 2

**Theorem** [C-F, 2005]

Let \( n = 2 \) and suppose \( p(X) = 0 \) in \( M_d(\beta) \) for some \( p \in \mathcal{P}_2 \). Then \( \beta^{(2d)} \) has a representing measure (necessarily supported in \( \mathbb{Z}_p \)) \iff \( M_d(\beta) \) is positive and recursively generated, and \( \text{rank } M_d(\beta) \leq \text{card } \mathcal{V}_\beta \). In this case, either \( M_d(\beta) \) has a flat extension \( M_{d+1} \), or \( M_d(\beta) \) has a positive extension \( M_{d+1} \), which in turn has a flat extension \( M_{d+2} \).

**Note**

(i) This result solves the bivariate quartic moment problem \((n = 2, \beta \equiv \beta^{(4)})\) in the case when \( M_2(\beta) \) is singular. For the case when \( M_2(\beta) \succ 0 \) we will use alternate methods based on approximation and convexity.

(ii) The above result does not extend to \( y = x^3 \) [F, 2008]
Approximation methods (with Jiawang Nie)

Let $\eta = \text{dim } \mathcal{P}_{2d}$, so $\beta^{(2d)} \in \mathbb{R}^{\eta}$.

$\mathcal{R}_{n,d} := \{ \beta \in \mathbb{R}^{\eta} : \beta \text{ has a } K - \text{representing measure} \}$, convex cone

$S_{n,d} := \{ \beta \in \mathbb{R}^{\eta} : L_{\beta} \text{ is } K - \text{positive} \}$, convex cone

**Theorem** [F-Nie, 2009]

$S_{n,d} = \overline{\mathcal{R}_{n,d}}$. 
Strict $K$-positivity and representing measures

$L_\beta$ is strictly $K$-positive if

$$p \in \mathcal{P}_{2d}, \ p|K \geq 0, \ p|K \neq 0 \implies L_\beta(p) > 0.$$ 

$K$ is determining if $p \in \mathcal{P}_{2d}, \ p|K \equiv 0 \implies p \equiv 0.$

**Theorem [F-Nie, 2009]**

If $K$ is determining and $L_\beta$ is strictly $K$-positive, then $\beta$ has a $K$-representing measure.

**Proof.**

The hypotheses imply that

$$\beta \in \text{interior}(S_{n,d}) = \text{interior}(\text{closure}(\mathcal{R}_{n,d}))$$

$$= \text{interior}(\mathcal{R}_{n,d}) \subseteq \mathcal{R}_{n,d}.$$
Bivariate quartic moment problem
Consider the second case of Hilbert’s theorem, when $n = 2$, $d = 4$, and consider the corresponding moment problem for $\beta^{(4)}$. For $M_2(\beta)$ singular, the problem was solved by [Curto-F, 2005] (above).

Theorem [F-Jiawang Nie, 2009]
Let $n = 2$. If $M_2(\beta) \succ 0$, then $\beta$ has a representing measure.

Proof.
Let $K = \mathbb{R}^2$, determining. Since $M_2(\beta) \succ 0$, Hilbert’s theorem implies that $L_\beta$ is strictly $K$-positive: If $p|\mathbb{R}^2 \geq 0$, $p \not\equiv 0$, then $p = \sum p_i^2$ (with some $p_i \not\equiv 0$), so $L_\beta(p) = \sum \langle M_2(\beta)\hat{p}_i, \hat{p}_i \rangle > 0$. Apply the previous theorem. □
Lasserre’s method for polynomial optimization

For simplicity, we consider the polynomial optimization problem for $K_Q = \mathbb{R}^n$, i.e., $Q = \{q_0 \equiv 1\}$. Let $p \in \mathbb{R}[x_1, \ldots, x_n]$. For $2t \geq \deg p$, the $t$-th Lasserre “moment relaxation” for $p_\star \equiv \inf_{x \in \mathbb{R}^n} p(x)$ is defined by

$$p_t := \inf \{ L_\beta(p) : \beta \equiv \beta^{(2t)}, \beta_0 = 1, M_t(\beta) \succeq 0 \}.$$

Then $p_t \leq p_\star$, and for $t' \geq t$, $p_{t'} \geq p_t$; thus, $\{p_t\}$ is convergent, and $p_{\star}^{\text{mom}} \equiv \lim_{t \to \infty} p_t \leq p_\star$. In general, for fixed $t$, $p_t$ is not necessarily attained at any $\beta$. Assuming that the infimum is attained, at some optimal sequence $\beta \equiv \beta^{\{t\}}$, we seek criteria so that $L_\beta(p) = p_\star$, so that we have finite convergence of $\{p_s\}$ at stage $t$. 
**Lasserre’s stopping criterion**

Assume at stage $t$ that $\beta \equiv \beta^t$ has a representing measure $\mu$. Then

$$p_* = p_* \beta_0 = p_* \int 1 d\mu \leq \int p d\mu = L_\beta(p) = p_t \leq p_*,\$$

so we have convergence at stage $t$. Although the existence of a representing measure for $\beta^t$ is difficult to ascertain in general, Lasserre focuses on the easy-to-check case when $M_t(\beta)$ is flat, i.e., $\text{rank } M_t(\beta) = \text{rank } M_{t-1}(\beta)$. In this case, $\beta$ has a $\text{rank } M_t$-atomic representing measure, and the atoms are the global minimizers for $p$.

Can we find a more general, but still concrete, stopping criterion?
A more general stopping criterion

**Theorem** [LF-Jiawang Nie, 2010]

Let \( \beta \equiv \beta^t \). If \( L_\beta \) is positive, then \( p_t = p_* \).

Of course, in general, positivity for \( L_\beta \) is very difficult to check. In current work we are studying a class for which positivity is clear. Let \( F_d := \{ \beta \equiv \beta^{(2d)} : M_d(y) \succeq 0 \text{ is flat} \} \) (a subset of \( \mathbb{R}^\rho \), where \( \rho \equiv \rho_{2d} = \dim \mathcal{P}_{2d} \)). Consider \( \overline{F_d} \), the closure. If \( \beta = \lim_{k \to \infty} \beta[k] \), with each \( M_d(\beta[k]) \) positive and flat, then each \( L_{\beta[k]} \) is positive, so \( L_\beta \) is positive. Thus, \( M_d(\beta) \succeq 0 \), and

\[
\text{rank } M_d(\beta) \leq \liminf_{k \to \infty} \text{rank } M_d(\beta[k]) = \liminf_{k \to \infty} \text{rank } M_{d-1}(\beta[k]) \leq \rho_{d-1}.
\]

If \( M_d(\beta) \succeq 0 \) and \( \text{rank } M_d(\beta) \leq \rho_{d-1} \), does \( \beta \) belong to \( \overline{F_d} \)?
On limits of positive flat moment matrices

**Theorem** [F-Nie, 2010]

Let $n = 1$, or $d = 1$, or $n = d = 2$. If $M_d(\beta) \succeq 0$ and \text{rank} \ $M_d(\beta) \leq \rho_{d-1}$, then $\beta \in \overline{F_d}$. 