Completely Positive Maps in Quantum Information

Yiu Tung Poon
Department of Mathematics
Iowa State University
Ames, Iowa, USA

INFAS
November 14, 2009

Joint work with Chi-Kwong Li.

This research was partially supported by an NSF grant.

$\mathcal{H}, \ \mathcal{K}$: Hilbert space.

Basic notation and definitions

$\mathcal{H}, \ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H}, \mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

Basic notation and definitions

$\mathcal{H},\ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H},\mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim \mathcal{H} = n$, take $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{B}(\mathcal{H}) = M_n$,

Basic notation and definitions

$\mathcal{H}, \ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H}, \mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim \mathcal{H} = n$, take $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{B}(\mathcal{H}) = M_n$, the set of $n \times n$ complex matrices.

Basic notation and definitions

$\mathcal{H},\ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H},\mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim\mathcal{H}=n$, take $\mathcal{H}=\mathbb{C}^n$, and $\mathcal{B}(\mathcal{H})=M_n$, the set of $n\times n$ complex matrices. Similarly, take $(\mathcal{B}(\mathcal{H},\mathcal{K}))=M_{m,\ n}$ when $\dim\mathcal{K}=m$.

Basic notation and definitions

$\mathcal{H},\ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H},\mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim \mathcal{H} = n$, take $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{B}(\mathcal{H}) = M_n$, the set of $n \times n$ complex matrices. Similarly, take $(\mathcal{B}(\mathcal{H},\mathcal{K})) = M_{m,\ n}$ when $\dim \mathcal{K} = m$.

$\mathcal{B}(\mathcal{H})_+$: Positive semi-definite operators in $\mathcal{B}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H})_+$, write $A \geq 0$.

Basic notation and definitions

$\mathcal{H},\ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H},\mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim \mathcal{H} = n$, take $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{B}(\mathcal{H}) = M_n$, the set of $n \times n$ complex matrices. Similarly, take $(\mathcal{B}(\mathcal{H},\mathcal{K})) = M_{m,\ n}$ when $\dim \mathcal{K} = m$.

$\mathcal{B}(\mathcal{H})_+$: Positive semi-definite operators in $\mathcal{B}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H})_+$, write $A \geq 0$.

A C*-algebra is a normed closed $*$-subalgebra of some $\mathcal{B}(\mathcal{H})$.

Basic notation and definitions

$\mathcal{H},\ \mathcal{K}$: Hilbert space.

$\mathcal{B}(\mathcal{H})$ $(\mathcal{B}(\mathcal{H},\mathcal{K}))$: Bounded Linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$).

When $\dim \mathcal{H} = n$, take $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{B}(\mathcal{H}) = M_n$, the set of $n \times n$ complex matrices. Similarly, take $(\mathcal{B}(\mathcal{H},\mathcal{K})) = M_{m,\ n}$ when $\dim \mathcal{K} = m$.

$\mathcal{B}(\mathcal{H})_+$: Positive semi-definite operators in $\mathcal{B}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H})_+$, write $A \geq 0$.

A C*-algebra is a normed closed $*$-subalgebra of some $\mathcal{B}(\mathcal{H})$.

A linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is said to be positive if $\Phi(A) \geq 0$ for every $A \geq 0$. $(\Phi \geq 0)$

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$,

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$, we have $\Phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^k)$ such that $\Phi_k([a_{ij}]) = [\Phi(a_{ij})]$.

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$, we have $\Phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^k)$ such that $\Phi_k([a_{ij}]) = [\Phi\,(a_{ij})]$.
- Φ is completely positive (CP)

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$, we have $\Phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^k)$ such that $\Phi_k([a_{ij}]) = [\Phi(a_{ij})]$.
- Φ is completely positive (CP) if $\Phi_k \geq 0$ for all $k \geq 1$.

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$, we have $\Phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^k)$ such that $\Phi_k([a_{ij}]) = [\Phi\,(a_{ij})]$.
- Φ is completely positive (CP) if $\Phi_k \geq 0$ for all $k \geq 1$.
- Define $\Phi : M_2 \to M_2$ by $\Phi(A) = A^t$. Then $\Phi_1 \geq 0$ but $\Phi_2 \not\geq 0$

Completely positive maps

- Let $\mathcal{A}$ be a C*-algebra. Suppose we have a linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for every $k \geq 1$, we have $\Phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^k)$ such that $\Phi_k([a_{ij}]) = [\Phi(a_{ij})]$.
- Φ is completely positive (CP) if $\Phi_k \geq 0$ for all $k \geq 1$.
- Define $\Phi : M_2 \to M_2$ by $\Phi(A) = A^t$. Then $\Phi_1 \geq 0$ but $\Phi_2 \not\geq 0$
-

$$\Phi_2\left(\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}\right) = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^*\pi(A)V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^*\pi(A)V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

We may assume that $\pi(I) = I_{\mathcal{K}}$.

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^*\pi(A)V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

We may assume that $\pi(I) = I_{\mathcal{K}}$. Φ is unital if and only if V is an isometry,

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^* \pi(A) V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

We may assume that $\pi(I) = I_{\mathcal{K}}$. Φ is unital if and only if V is an isometry, i.e. $V^*V = I_{\mathcal{H}}$.

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^*\pi(A)V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

We may assume that $\pi(I) = I_{\mathcal{K}}$. Φ is unital if and only if V is an isometry, i.e. $V^*V = I_{\mathcal{H}}$. In this case, we say that $\Phi(A)$ is a compression of $\pi(A)$,

Stinespring's Theorem

Stinespring's Theorem 1955

Let $\mathcal{A}$ be a C*-algebra with a unit, let $\mathcal{H}$ be a Hilbert space, and let Φ be a linear function from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Then a necessary and sufficient condition for Φ to be completely positive is that Φ have the form

$$\Phi(A) = V^*\pi(A)V$$

for all $A \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and π is a $*$-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.

We may assume that $\pi(I) = I_{\mathcal{K}}$. Φ is unital if and only if V is an isometry, i.e. $V^*V = I_{\mathcal{H}}$. In this case, we say that $\Phi(A)$ is a compression of $\pi(A)$, or $\pi(A)$ is a dilation of $\Phi(A)$.

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \qquad (1)$$

where $F_1, \ldots, F_r \in M_{n,m}$.

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \qquad (1)$$

where $F_1, \dots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \tag{1}$$

where $F_1, \ldots, F_r \in M_{n,\,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps,

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \tag{1}$$

where $F_1, \ldots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\operatorname{tr} \Phi(A) = \operatorname{tr} A$.

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \tag{1}$$

where $F_1, \dots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\mathrm{tr}\, \Phi(A) = \mathrm{tr}\, A$. $\mathrm{tr}\, (a_{i\,j}) = \sum_i a_{i\,i}$.

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \qquad (1)$$

where $F_1, \dots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\mathrm{tr}\, \Phi(A) = \mathrm{tr}\, A$. $\mathrm{tr}\ (a_{i\,j}) = \sum_i a_{i\,i}$.
- Φ is trace preserving if and only if $\sum_{j=1}^{r} F_j F_j^* = I_n$.

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \tag{1}$$

where $F_1, \ldots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\mathrm{tr}\,\Phi(A) = \mathrm{tr}\,A$. $\mathrm{tr}\,(a_{i\,j}) = \sum_i a_{i\,i}$.
- Φ is trace preserving if and only if $\sum_{j=1}^{r} F_j F_j^* = I_n$.
- Φ is said to have Choi's rank $\leq r$ if it can be represented in (1).

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \qquad (1)$$

where $F_1, \dots, F_r \in M_{n,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\operatorname{tr} \Phi(A) = \operatorname{tr} A$. $\operatorname{tr}\,(a_{i\,j}) = \sum_i a_{i\,i}$.
- Φ is trace preserving if and only if $\sum_{j=1}^{r} F_j F_j^* = I_n$.
- Φ is said to have Choi's rank $\leq r$ if it can be represented in (1). r can be interpreted as the dimension of the external environment E

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \tag{1}$$

where $F_1, \ldots, F_r \in M_{n,\,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\mathrm{tr}\,\Phi(A) = \mathrm{tr}\,A$. $\mathrm{tr}\,(a_{i\,j}) = \sum_i a_{i\,i}$.
- Φ is trace preserving if and only if $\sum_{j=1}^{r} F_j F_j^* = I_n$.
- Φ is said to have Choi's rank $\leq r$ if it can be represented in (1). r can be interpreted as the dimension of the external environment E required to couple with the principal system

Choi's Theorems

Choi's Theorem 1975

Let $\Phi : M_n \to M_m$. Then Φ is completely positive if and only if Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j, \qquad (1)$$

where $F_1, \ldots, F_r \in M_{n,\,m}$.

- Φ is unital if and only if $\sum_{j=1}^{r} F_j^* F_j = I_m$.
- In quantum information, quantum channels are trace preserving completely positive maps, i.e. $\mathrm{tr}\,\Phi(A) = \mathrm{tr}\,A$. $\mathrm{tr}\,(a_{i\,j}) = \sum_i a_{i\,i}$.
- Φ is trace preserving if and only if $\sum_{j=1}^{r} F_j F_j^* = I_n$.
- Φ is said to have Choi's rank $\leq r$ if it can be represented in (1). r can be interpreted as the dimension of the external environment E required to couple with the principal system so that the CP map can be evaluated from the composite system.

Duality

- $\langle A, B\rangle = \operatorname{tr} AB^*$ defines an inner product on $M_{n,m}$.

Duality

- $\langle A, B\rangle = \operatorname{tr} AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear.

Duality

- $\langle A, B\rangle = \mathrm{tr}\, AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear. Then the dual map $\Phi^* : M_m \to M_n$ is defined by $\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle$ for all $A \in M_n,\ B \in M_m$.

Duality

- $\langle A, B \rangle = \operatorname{tr} AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear. Then the dual map $\Phi^* : M_m \to M_n$ is defined by $\langle \Phi(A), B \rangle = \langle A, \Phi^*(B) \rangle$ for all $A \in M_n,\ B \in M_m$.
- If

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j,$$

Duality

- $\langle A, B\rangle = \operatorname{tr} AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear. Then the dual map $\Phi^* : M_m \to M_n$ is defined by $\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle$ for all $A \in M_n,\ B \in M_m$.
- If
$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j,$$
then
$$\Phi^*(B) = \sum_{j=1}^{r} F_j B F_j^*.$$

Duality

- $\langle A, B\rangle = \operatorname{tr} AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear. Then the dual map $\Phi^* : M_m \to M_n$ is defined by $\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle$ for all $A \in M_n,\ B \in M_m$.
- If
$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j,$$
then
$$\Phi^*(B) = \sum_{j=1}^{r} F_j B F_j^*.$$
- A completely positive map $\Phi : M_n \to M_m$ is unital if and only if Φ^* is trace preserving.

Duality

- $\langle A, B\rangle = \mathrm{tr}\, AB^*$ defines an inner product on $M_{n,m}$. Suppose $\Phi : M_n \to M_m$ is linear. Then the dual map $\Phi^* : M_m \to M_n$ is defined by $\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle$ for all $A \in M_n,\ B \in M_m$.
- If
$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j,$$
then
$$\Phi^*(B) = \sum_{j=1}^{r} F_j B F_j^*.$$
- A completely positive map $\Phi : M_n \to M_m$ is unital if and only if Φ^* is trace preserving.
- Can we use this duality to deduce results in trace preserving completely positive maps from those in unital completely positive maps and vice versa?

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices.

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.
5. Given $A_1, \ldots, A_k \in H_n$ and $B_1, \ldots, B_k \in H_m$, under what condition can we have a completely positive map Φ such that $\Phi(A_i) = B_i$?

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.
5. Given $A_1, \ldots, A_k \in H_n$ and $B_1, \ldots, B_k \in H_m$, under what condition can we have a completely positive map Φ such that $\Phi(A_i) = B_i$? If Φ exists, can we choose Φ with additional properties?

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.
5. Given $A_1, \ldots, A_k \in H_n$ and $B_1, \ldots, B_k \in H_m$, under what condition can we have a completely positive map Φ such that $\Phi(A_i) = B_i$? If Φ exists, can we choose Φ with additional properties? E.g. unital,

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.
5. Given $A_1, \ldots, A_k \in H_n$ and $B_1, \ldots, B_k \in H_m$, under what condition can we have a completely positive map Φ such that $\Phi(A_i) = B_i$? If Φ exists, can we choose Φ with additional properties? E.g. unital, trace preserving,

General problems

1. Given $A \in M_n$ and $B \in M_m$, when is there a completely positive map $\Phi(A) = B$?
2. What about $\Phi(A_i) = B_i$ for $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$?
3. Deduce properties of Φ based on the information of $\Phi(A)$ for some special matrices A.
4. Let $H_n = \{A \in M_n : A = A^*\}$ be the set of $n \times n$ Hermitian matrices. Φ is determined by its action on H_n.
5. Given $A_1, \ldots, A_k \in H_n$ and $B_1, \ldots, B_k \in H_m$, under what condition can we have a completely positive map Φ such that $\Phi(A_i) = B_i$? If Φ exists, can we choose Φ with additional properties? E.g. unital, trace preserving, Choi's rank $\leq r$.

Why study?

Why study?

- It is a general interpolation problem for completely positive maps.

Why study?

- It is a general interpolation problem for completely positive maps.
- It is related to dilation/compression problems of operators.

Why study?

- It is a general interpolation problem for completely positive maps.
- It is related to dilation/compression problems of operators.
- In quantum information, quantum operations and channels are completely positive maps.

Why study?

- It is a general interpolation problem for completely positive maps.
- It is related to dilation/compression problems of operators.
- In quantum information, quantum operations and channels are completely positive maps. The study is relevant to problems in quantum tomography, quantum control and quantum error correction.

Why study?

- It is a general interpolation problem for completely positive maps.
- It is related to dilation/compression problems of operators.
- In quantum information, quantum operations and channels are completely positive maps. The study is relevant to problems in quantum tomography, quantum control and quantum error correction.
- One often tries to deduce the properties of Φ through the study of $\Phi(A)$ for some special A.

Why study?

- It is a general interpolation problem for completely positive maps.
- It is related to dilation/compression problems of operators.
- In quantum information, quantum operations and channels are completely positive maps. The study is relevant to problems in quantum tomography, quantum control and quantum error correction.
- One often tries to deduce the properties of Φ through the study of $\Phi(A)$ for some special A.
- The study is related to other topics such as matrix inequalities (majorization), unitary orbits (algebraic, analytic and geometric properties), algebraic combinatorics etc.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \textit{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2)

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$. Here, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$. Here, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. Let $\lambda(A) = (\lambda_1(A) \geq \cdots \geq \lambda_n(A))$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$. Here, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. Let $\lambda(A) = (\lambda_1(A) \geq \cdots \geq \lambda_n(A))$.

Majorization

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, with $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$ for $1 \leq i \leq n-1$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$. Here, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. Let $\lambda(A) = (\lambda_1(A) \geq \cdots \geq \lambda_n(A))$.

Majorization

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, with $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$ for $1 \leq i \leq n-1$. $\mathbf{b}$ is said to be majorized by $\mathbf{a}$, $\mathbf{b} \prec \mathbf{a}$ if

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. Then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$ if and only if there are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \text{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A). \tag{2}$$

Moreover, there is $\gamma = \gamma_1 = \gamma_2 > 0$ satisfies (2) if and only if there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(I_n) = \gamma I_m$ and $\Phi(A) = B$. Here, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. Let $\lambda(A) = (\lambda_1(A) \geq \cdots \geq \lambda_n(A))$.

Majorization

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, with $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$ for $1 \leq i \leq n-1$. $\mathbf{b}$ is said to be majorized by $\mathbf{a}$, $\mathbf{b} \prec \mathbf{a}$ if $\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i$ for all $1 \leq k \leq n-1$ and $\sum_{i=1}^n b_i = \sum_{i=1}^n a_i$.

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n - k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n - k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n - k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n-k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

(c) There is an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n-k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

(c) There is an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$. $D = (d_{i\,j})$ with $d_{i\,j} \geq 0$,

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n - k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

(c) There is an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$. $D = (d_{i\,j})$ with $d_{i\,j} \geq 0$, $\sum_{j=1}^{n} d_{i\,j} = 1$ for all $1 \leq i \leq n$.

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n-k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

(c) There is an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$. $D = (d_{i\,j})$ with $d_{i\,j} \geq 0$, $\sum_{j=1}^{n} d_{i\,j} = 1$ for all $1 \leq i \leq n$.

(d) There is an $n \times m$ row stochastic matrix D with the first k rows all equal and the last $n-k$ rows all equal such that $\lambda(B) = \lambda(A)D$.

Some results for $k = 1$

Theorem

Suppose $A \in H_n$ has k non-negative eigenvalues and $n-k$ negative eigenvalues, and $B \in H_m$. The following conditions are equivalent.

(a) There is a trace preserving completely positive linear map $\Phi : M_n \to M_m$ such that $\phi(A) = B$.

(b) $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-) \in \mathbb{R}^m$ where

$$a_+ = \sum_{j=1}^{k} \lambda_j(A) \quad \text{and} \quad a_- = \sum_{j=k+1}^{n} \lambda_j(A).$$

(c) There is an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$. $D = (d_{i\,j})$ with $d_{i\,j} \geq 0$, $\sum_{j=1}^{n} d_{i\,j} = 1$ for all $1 \leq i \leq n$.

(d) There is an $n \times m$ row stochastic matrix D with the first k rows all equal and the last $n-k$ rows all equal such that $\lambda(B) = \lambda(A)D$.

Remark For density matrices, $(A,\ B \geq 0, \mathrm{tr}\, A = \mathrm{tr}\, B = 1)$, we have $\lambda(B) \prec (1, 0, \ldots, 0)$ and condition (b) holds.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.
- There is an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.
- There is an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Remark The condition may fail even if A and B are density matrices.

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.
- There is an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Remark The condition may fail even if A and B are density matrices.

Question

Suppose there is a unital completely positive map sending A to B,

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.
- There is an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Remark The condition may fail even if A and B are density matrices.

Question

Suppose there is a unital completely positive map sending A to B, **and** a trace preserving completely positive map sending A to B.

Some results for $k = 1$

Theorem

Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

- There is a unital completely positive linear map Φ such that $\Phi(A) = B$.
- $\lambda_n(A) \leq \lambda_j(B) \leq \lambda_1(A)$ for all $j = 1, \ldots, m$.
- There is an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Remark The condition may fail even if A and B are density matrices.

Question

Suppose there is a unital completely positive map sending A to B, **and** a trace preserving completely positive map sending A to B.
Is there a unital and trace preserving completely positive map sending A to B?

Example

Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$.

Some results for $k = 1$

Example

Suppose $A = \operatorname{diag}(4, 1, 1, 0)$ and $B = \operatorname{diag}(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B,

Some results for $k = 1$

Example

Suppose $A = \operatorname{diag}(4,1,1,0)$ and $B = \operatorname{diag}(3,3,0,0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B.

Some results for $k = 1$

Example

Suppose $A = \operatorname{diag}(4, 1, 1, 0)$ and $B = \operatorname{diag}(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \operatorname{diag}(3, 0, 0, -1)$ and $B_1 = B - I_4 = \operatorname{diag}(2, 2, -1, -1)$.

Some results for $k = 1$

Example

Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \mathrm{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$.
Then there is no trace preserving CP map sending A_1 to B_1.

Example

Suppose $A = \mathrm{diag}\,(4,1,1,0)$ and $B = \mathrm{diag}\,(3,3,0,0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \mathrm{diag}\,(3,0,0,-1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2,2,-1,-1)$.
Then there is no trace preserving CP map sending A_1 to B_1. Hence, there is no unital trace preserving CP map sending A to B.

Example

Suppose $A = \text{diag}\,(4, 1, 1, 0)$ and $B = \text{diag}\,(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \text{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \text{diag}\,(2, 2, -1, -1)$.
Then there is no trace preserving CP map sending A_1 to B_1. Hence, there is no unital trace preserving CP map sending A to B.

Randomized unitary channel

- Suppose $\Phi : M_n \to M_n$ is given by $\Phi(A) = \sum_{j=1}^{r} t_i U_j^* A U_j$, where $U_1, \ldots, U_r$ are unitaries and $t_i \geq 0$, with $\sum_{j=1}^{r} t_i = 1$.

Some results for $k = 1$

Example

Suppose $A = \operatorname{diag}(4,1,1,0)$ and $B = \operatorname{diag}(3,3,0,0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \operatorname{diag}(3,0,0,-1)$ and $B_1 = B - I_4 = \operatorname{diag}(2,2,-1,-1)$.
Then there is no trace preserving CP map sending A_1 to B_1. Hence, there is no unital trace preserving CP map sending A to B.

Randomized unitary channel

- Suppose $\Phi : M_n \to M_n$ is given by $\Phi(A) = \sum_{j=1}^{r} t_i U_j^* A U_j$, where $U_1, \ldots, U_r$ are unitaries and $t_i \geq 0$, with $\sum_{j=1}^{r} t_i = 1$. Then Φ is known as a randomized unitary channel in quantum information.

Some results for $k = 1$

Example

Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let
$A_1 = A - I_4 = \mathrm{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$.
Then there is no trace preserving CP map sending A_1 to B_1. Hence, there is no unital trace preserving CP map sending A to B.

Randomized unitary channel

- Suppose $\Phi : M_n \to M_n$ is given by $\Phi(A) = \sum_{j=1}^{r} t_i U_j^* A U_j$, where $U_1, \ldots, U_r$ are unitaries and $t_i \geq 0$, with $\sum_{j=1}^{r} t_i = 1$. Then Φ is known as a randomized unitary channel in quantum information. In this case, Φ is a unital and trace preserving CP map.

Example

Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$. Then there is a trace preserving CP map sending A to B, and also a unital CP map sending A to B. Let $A_1 = A - I_4 = \mathrm{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$. Then there is no trace preserving CP map sending A_1 to B_1. Hence, there is no unital trace preserving CP map sending A to B.

Randomized unitary channel

- Suppose $\Phi : M_n \to M_n$ is given by $\Phi(A) = \sum_{j=1}^{r} t_i U_j^* A U_j$, where $U_1, \ldots, U_r$ are unitaries and $t_i \geq 0$, with $\sum_{j=1}^{r} t_i = 1$. Then Φ is known as a randomized unitary channel in quantum information. In this case, Φ is a unital and trace preserving CP map.
- For $n \geq 3$, there exists a unital and trace preserving CP map $\Phi : M_n \to M_n$ which is not a randomized unitary channel.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.
- For each $t \in \mathbb{R}$, there exists a trace preserving CP map Φ_t such that $\Phi_t(A - tI) = B - tI$.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.
- For each $t \in \mathbb{R}$, there exists a trace preserving CP map Φ_t such that $\Phi_t(A - tI) = B - tI$.
- $\lambda(B) = \operatorname{diag}(U^* A U)$ for some unitary U.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.
- For each $t \in \mathbb{R}$, there exists a trace preserving CP map Φ_t such that $\Phi_t(A - tI) = B - tI$.
- $\lambda(B) = \mathrm{diag}\,(U^* AU)$ for some unitary U.
- $\lambda(B) \prec \lambda(A)$

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.
- For each $t \in \mathbb{R}$, there exists a trace preserving CP map Φ_t such that $\Phi_t(A - tI) = B - tI$.
- $\lambda(B) = \mathrm{diag}\,(U^* AU)$ for some unitary U.
- $\lambda(B) \prec \lambda(A)$
- There is an $n \times n$ doubly stochastic (d.s.) matrix D such that $\lambda(B) = \lambda(A)D$.

Theorem

Let $A, B \in H_n$. The following conditions are equivalent.

- $\Phi(A) = B$ for a unital trace preserving CP map Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ.
- $\Phi(A) = B$ for a randomized unitary channel Φ, with all $t_i = 1/n$.
- For each $t \in \mathbb{R}$, there exists a trace preserving CP map Φ_t such that $\Phi_t(A - tI) = B - tI$.
- $\lambda(B) = \operatorname{diag}(U^* A U)$ for some unitary U.
- $\lambda(B) \prec \lambda(A)$
- There is an $n \times n$ doubly stochastic (d.s.) matrix D such that $\lambda(B) = \lambda(A) D$.

D is d.s. if it is both row stochastic and column stochastic.

Restricted rank

Restricted rank

We will use $CP_r(n, m)$ to denote CP maps with Choi's rank $\leq r$

Restricted rank

We will use $CP_r(n,m)$ to denote CP maps with Choi's rank $\leq r$ and $CP(n,m) = \bigcup_{r\geq 1} CP_r(n,m)$.

Restricted rank

We will use $CP_r(n,m)$ to denote CP maps with Choi's rank $\leq r$ and $CP(n,m) = \bigcup_{r\geq 1} CP_r(n,m)$.

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

Restricted rank

We will use $CP_r(n,m)$ to denote CP maps with Choi's rank $\leq r$ and $CP(n,m) = \bigcup_{r\geq 1} CP_r(n,m)$.

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

Restricted rank

We will use $CP_r(n,m)$ to denote CP maps with Choi's rank $\leq r$ and $CP(n,m) = \bigcup_{r\geq 1} CP_r(n,m)$.

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.
- There exists an $(nr) \times m$ matrix F such that $B = F^*(A \otimes I_r)F$.

Restricted rank

We will use $CP_r(n,m)$ to denote CP maps with Choi's rank $\leq r$ and $CP(n,m) = \bigcup_{r\geq 1} CP_r(n,m)$.

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.
- There exists an $(nr) \times m$ matrix F such that $B = F^*(A \otimes I_r)F$.
- The number of positive (negative) eigenvalues of $A \otimes I_r$ is not less than the number of positive (negative) eigenvalues of B.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists a unital $\Phi \in CP_r(n, m)$ such that $\Phi(A) = B$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists a unital $\Phi \in CP_r(n, m)$ such that $\Phi(A) = B$.
- B is a compression of $A \otimes I_r$, i.e. $B = F^*(A \otimes I_r)F$ for some $(nr) \times m$ F such that $F^*F = I_m$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists a unital $\Phi \in CP_r(n, m)$ such that $\Phi(A) = B$.
- B is a compression of $A \otimes I_r$, i.e. $B = F^*(A \otimes I_r)F$ for some $(nr) \times m$ F such that $F^*F = I_m$.
- The eigenvalues of $A \otimes I_r$ interlace those of B, i.e.,

$$\lambda_j(A \otimes I_r) \geq \lambda_j(B) \quad \text{and} \quad \lambda_{m+1-j}(B) \geq \lambda_{n+1-j}(A \otimes I_r)$$

  for $1 \leq j \leq m$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$. Suppose $nr \geq m$. The following conditions are equivalent.

- There exists a unital $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.
- B is a compression of $A \otimes I_r$, i.e. $B = F^*(A \otimes I_r)F$ for some $(nr) \times m$ F such that $F^*F = I_m$.
- The eigenvalues of $A \otimes I_r$ interlace those of B, i.e.,

  $$\lambda_j(A \otimes I_r) \geq \lambda_j(B) \quad \text{and} \quad \lambda_{m+1-j}(B) \geq \lambda_{n+1-j}(A \otimes I_r)$$

  for $1 \leq j \leq m$.

Remark Even if A, B are density matrices, the roles of A and B are not symmetric in the last two theorems.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n, m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^{r} C_{jj}$.

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n, m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^{r} C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \ldots, C_r \in H_m$,

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^{r} C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \ldots, C_r \in H_m$, and unitary matrices $U_1, \ldots, U_r \in M_n$

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^r C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \ldots, C_r \in H_m$, and unitary matrices $U_1, \ldots, U_r \in M_n$ such that $A = \sum_{j=1}^r U_j^* \left(C_j \oplus O_{n-m}\right) U_j$ and $B = \sum_{j=1}^r C_j$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^{r} C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \ldots, C_r \in H_m$, and unitary matrices $U_1, \ldots, U_r \in M_n$ such that $A = \sum_{j=1}^{r} U_j^* \left(C_j \oplus O_{n-m}\right) U_j$ and $B = \sum_{j=1}^{r} C_j$.

(d) There exists a trace preserving $\Psi \in CP_r(m,n)$ such that $\Psi(B) = A$.

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^r C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \ldots, C_r \in H_m$, and unitary matrices $U_1, \ldots, U_r \in M_n$ such that $A = \sum_{j=1}^r U_j^* \left(C_j \oplus O_{n-m}\right) U_j$ and $B = \sum_{j=1}^r C_j$.

(d) There exists a trace preserving $\Psi \in CP_r(m,n)$ such that $\Psi(B) = A$.

C_j can be choose in the unitary orbit of some $C \in H_m$.

Restricted rank

Theorem

Let $A \in H_n$ and $B \in H_m$ be positive semidefinite, with $mr \geq n$. The following conditions are equivalent.

(a) There exists a trace preserving $\Phi \in CP_r(n,m)$ such that $\Phi(A) = B$.

(b) There exists a unitary $U \in M_{mr}$ such that $U^*(A \oplus O_{mr-n})U = (C_{i\,j})$, $C_{i\,j} \in M_m$ with $B = \sum_{j=1}^r C_{jj}$.

(c) There exist positive semidefinite matrices $C_1, \dots, C_r \in H_m$, and unitary matrices $U_1, \dots, U_r \in M_n$ such that $A = \sum_{j=1}^r U_j^* \,(C_j \oplus O_{n-m})\, U_j$ and $B = \sum_{j=1}^r C_j$.

(d) There exists a trace preserving $\Psi \in CP_r(m,n)$ such that $\Psi(B) = A$.

C_j can be choose in the unitary orbit of some $C \in H_m$.

Remark Conditions (c) can be expressed in terms of eigenvalue inequalities using Littlewood-Richardson rules in Schubert calculus.

Results and questions on multiple matrices

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

Results and questions on multiple matrices

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Question What about non-commuting families?

Results and questions on multiple matrices

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Question What about non-commuting families? Restricted Choi's rank?

Results and questions on multiple matrices

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Question What about non-commuting families? Restricted Choi's rank?

Consider the case when $k = 2$. Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$.

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Question What about non-commuting families? Restricted Choi's rank?

Consider the case when $k = 2$. Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2 \in M_n$ and $B = B_1 + iB_2$.

Results and questions on multiple matrices

Theorem

Suppose $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ are commuting families of matrices in M_n and M_m. Then there is a unital / trace preserving / unital and trace preserving completely positive linear maps Φ such that

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k$$

if and only if there is an $n \times m$ column / row / doubly stochastic matrix D such that $\lambda(B_j) = \lambda(A_j)D$ for $j = 1, \ldots, k$.

Question What about non-commuting families? Restricted Choi's rank?

Consider the case when $k = 2$. Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2 \in M_n$ and $B = B_1 + iB_2$. We will study $B = \Phi(A)$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is
$W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is
$W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000, 2001) Let $A_1 \in M_2$ and $A = A_1$ or $A = A_1 \oplus [a]$.

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000, 2001) Let $A_1 \in M_2$ and $A = A_1$ or $A = A_1 \oplus [a]$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is
$W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000, 2001) Let $A_1 \in M_2$ and $A = A_1$ or $A = A_1 \oplus [a]$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000, 2001) Let $A_1 \in M_2$ and $A = A_1$ or $A = A_1 \oplus [a]$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000) Let $A = \operatorname{diag}(1, -1, i, -i)$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Dilation and compression

Let $T \in \mathcal{B}(\mathcal{H})$. The numerical range of A is $W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|x\| = 1\}$.

- (Mirman, 1968) Let $A \in M_3$ be normal. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Ando, 1973) Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000, 2001) Let $A_1 \in M_2$ and $A = A_1$ or $A = A_1 \oplus [a]$. Then $B = \Phi(A)$ for a unital CP map Φ if and only if $W(B) \subseteq W(A)$.
- (Choi, Li, 2000) Let $A = \operatorname{diag}(1, -1, i, -i)$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. Then $W(B) \subseteq W(A)$ but there is no unital CP map Φ such that $B = \Phi(A)$.

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$.

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

- There exists a trace preserving $\Phi \in CP_1(n,m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if the B is unitary similar to $A \oplus O_{m-n}$.

Choi's rank = 1

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

- There exists a trace preserving $\Phi \in CP_1(n, m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if the B is unitary similar to $A \oplus O_{m-n}$.
- There exists a unital $\Phi \in CP_1(n, m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if B is a compression of A,

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

- There exists a trace preserving $\Phi \in CP_1(n, m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if the B is unitary similar to $A \oplus O_{m-n}$.
- There exists a unital $\Phi \in CP_1(n, m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if B is a compression of A, i.e. there is a unitary U such that B is a principal submatrix of U^*AU.

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

- There exists a trace preserving $\Phi \in CP_1(n,m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if the B is unitary similar to $A \oplus O_{m-n}$.
- There exists a unital $\Phi \in CP_1(n,m)$ such that $\Phi(A_j) = B_j$ for $j = 1, 2$ if and only if B is a compression of A, i.e. there is a unitary U such that B is a principal submatrix of U^*AU. The problem is unsolved for $n = 4$, $m = 2$

Choi's rank $= 1$

Suppose $A_1, A_2 \in H_n$ and $B_1, B_2 \in H_m$. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$.

- There exists a trace preserving $\Phi \in CP_1(n,m)$ such that $\Phi(A_j) = B_j$ for $j = 1,2$ if and only if the B is unitary similar to $A \oplus O_{m-n}$.
- There exists a unital $\Phi \in CP_1(n,m)$ such that $\Phi(A_j) = B_j$ for $j = 1,2$ if and only if B is a compression of A, i.e. there is a unitary U such that B is a principal submatrix of U^*AU. The problem is unsolved for $n = 4$, $m = 2$ even when A_1 and A_2 commute.

Restricted rank on multiple matrices

Theorem [Poon 1992]

Let $\Phi \in CP(n,m)$ be unital and $A_1, \ldots, A_k \in H_n$.

Restricted rank on multiple matrices

Theorem [Poon 1992]

Let $\Phi \in CP(n,m)$ be unital and $A_1, \ldots, A_k \in H_n$. If $1 \le r \le mn - 1$ and $m^2(k+1) - 1 < (r+1)^2 - \delta_{mn,\, r+1}$,

Restricted rank on multiple matrices

Theorem [Poon 1992]

Let $\Phi \in CP(n, m)$ be unital and $A_1, \ldots, A_k \in H_n$. If $1 \leq r \leq mn - 1$ and $m^2(k+1) - 1 < (r+1)^2 - \delta_{mn,\, r+1}$, then there exists a unital $\Psi \in CP_r(n\, m)$ such that for all $1 \leq i \leq k$,

$$\Psi(A_i) = \Phi(A_i)$$

Restricted rank on multiple matrices

Theorem [Poon 1992]

Let $\Phi \in CP(n, m)$ be unital and $A_1, \ldots, A_k \in H_n$. If $1 \leq r \leq mn - 1$ and $m^2(k+1) - 1 < (r+1)^2 - \delta_{mn,\, r+1}$, then there exists a unital $\Psi \in CP_r(n\, m)$ such that for all $1 \leq i \leq k$,

$$\Psi(A_i) = \Phi(A_i)$$

For example, if $n > 2$, $k = 3$ and $r = 2m - 1$, then the condition is satisfied.

Restricted rank on multiple matrices

Theorem [Poon 1992]

Let $\Phi \in CP(n,m)$ be unital and $A_1, \ldots, A_k \in H_n$. If $1 \le r \le mn-1$ and $m^2(k+1)-1 < (r+1)^2 - \delta_{mn,\,r+1}$, then there exists a unital $\Psi \in CP_r(n\,m)$ such that for all $1 \le i \le k$,

$$\Psi(A_i) = \Phi(A_i)$$

For example, if $n > 2$, $k = 3$ and $r = 2m-1$, then the condition is satisfied.

What about trace preserving completely positive maps?

Thank you for your attention!