Feynman's Operational Calculus

Background and a Survey of Current Research

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Outline

1. Background
   - Why do we need functions of operators?
   - How do we form functions of operators?
   - The General Formalism
2. Recent Results/Current Research
   - Algebraic Results
   - Analytic Results

Outline

Elementary DE/Quantum Mechanics

- Consider the DE $\frac{d\vec{x}}{dt} = A\vec{x}(t)$ with the initial condition $\vec{x}(0) = \vec{c}$ where A is a constant square matrix. The solution can be written as $\vec{x}(t) = \vec{c}e^{tA}$. Hence, we need to be able to form a function of the matrix (operator) A.
- Of course, in quantum mechanics, we also have need of forming exponential functions of operators such as $e^{-i(H_0+V)}$.

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Other Considerations

- Suppose that we consider the function $f(x, y) = xy$.
- Let $A, B \in \mathscr{L}(X)$ where X is a Banach space. Assume that $AB \neq BA$.
- What is $f(A, B)$?
  - One possibility is that we take $f(A, B) = AB$. But we could also take $f(A, B) = BA$.
  - We could also take $f(A, B) = \frac{1}{3}AB + \frac{2}{3}BA$ - There are many other possibilities.
  - There is an ambiguity present when forming functions of noncommuting operators. How can we specify what "version" we wish to work with?

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Feynman's "Rules"

Feynman's Heuristic Rules

1. Attach time indices to the operators in question, in order to specify the order of operation in products.
2. Form the desired function of the operators, just as if they were commuting.
3. Disentangle the result, that is, bring the expression to a sum of time-ordered expressions.

These "rules" come from Feynman's 1951 paper [Feyn51]

Attaching Time Indices

Suppose $A \in \mathscr{L}(X)$, X a Banach space.

- Feynman would typically attach time indices via

$$A = \frac{1}{t} \int_0^t A(s)\, ds$$

where we take $A(s) \equiv A$ for all $s \geq 0$.

- We will, following Johnson and Lapidus [DJL, JLMem, JLBook](also found in [JLNBook]) attach time indices using probability measures. Let $T > 0$. Let μ be a Borel probability measure on $[0, T]$. We can write

$$A = \int_{[0,T]} A(s)\, \mu(ds).$$

Disentangling

Feynman, in his 1951 paper [Feyn51] makes the following remark concerning the process of disentangling:

"The process is not always easy to perform and, in fact, is the central problem of this operator calculus." (p. 110)

Also found in the same paper is the remark:

"The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that the demonstrations can be easily supplied. Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author." (p. 108)

Basic Example 1

We return to the function $f(x,y) = xy$. Let $A, B \in \mathscr{L}(X)$ and associate Lebesgue measure on $[0,1]$ to each operator. We calculate $f(A,B)$ as follows:

$$
\begin{aligned}
& f_{\ell,\ell}(A,B) = \left\{ \int_0^1 A(s)\,ds \right\} \left\{ \int_0^1 B(s)\,ds \right\} \\
& = \int_0^1 \int_0^1 A(s_1) B(s_2)\, ds_1\, ds_2 \\
& = \int_{\{(s_1,s_2): s_1 < s_2\}} B(s_2) A(s_1)\, ds_1\, ds_2 + \int_{\{(s_1,s_2): s_2 < s_1\}} A(s_1) B(s_2)\, ds_1\, ds_2 \\
& = \frac{1}{2} BA + \frac{1}{2} AB
\end{aligned}
$$

Basic Example 2

We stay with the function $f(x,y) = xy$ and operators $A, B \in \mathscr{L}(X)$. However, we will associate the Dirac point mass δ_τ for $\tau \in (0,1)$ to B and keep Lebesgue measure with A. Then

$$f_{\ell,\delta_\tau}(A,B) = \left\{ \int_0^1 A(s_1)\, ds_1 \right\} \left\{ \int_0^1 B(s_2)\, \delta_\tau(ds_2) \right\}$$

$$= \left\{ \int_{[0,\tau)} A(s_1)\, ds_1 + \int_{(\tau,1]} A(s_1)\, ds_1 \right\} \left\{ \int_{\{\tau\}} B(s_2)\, \delta_\tau(ds_2) \right\}$$

$$= \left\{ \int_{\{\tau\}} B(s_2)\, \delta_\tau(ds_2) \right\} \left\{ \int_{[0,\tau)} A(s_1)\, ds_1 \right\} +$$

$$\left\{ \int_{(\tau,1]} A(s_1)\, ds_1 \right\} \left\{ \int_{\{\tau\}} B(s_2)\, \delta_\tau(ds_2) \right\} = \tau BA + (1-\tau)AB$$

Basic Example 3

We continue with $f(x,y) = xy$ and the operators $A, B \in \mathscr{L}(X)$. Associate to A a probability measure μ on $[0,1]$ with support in $(0,a)$ and associate to B a probability measure ν on $[0,1]$ with support in $(a,1)$. We have

$$\begin{aligned} f_{\mu,\nu}(A,B) &= \left\{ \int_{[0,1]} A(s)\mu(ds) \right\} \left\{ \int_{[0,1]} B(s)\nu(ds) \right\} \\ &= \left\{ \int_{[a,1]} B(s)\nu(ds) \right\} \left\{ \int_{[0,a]} A(s)\mu(ds) \right\} \\ &= BA \end{aligned}$$

Comment

It is clear that the time-ordering measures that we use to attach time indices to operators play a critical role in forming functions of noncommuting operators.

Outline

The Commutative Banach Algebra $\mathbb{A}$

1. Let $A_1, \ldots, A_n \in \mathscr{L}(X)$, X a Banach space. Define the commutative Banach algebra $\mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$ to be the family of all functions $f(z_1, \ldots, z_n)$ of whose Taylor series converge on the closed polydisk $\{(z_1, \ldots, z_n) : |z_j| \leq \|A_j\|, j = 1, \ldots, n\}$. Such functions are analytic on the open polydisk with radii $\|A_j\|$, $j = 1, \ldots, n$, and continuous on the boundary of the closed polydisk.
2. If we define $\|f\|_{\mathbb{A}} := \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n}$, $\mathbb{A}$ is a commutative Banach algebra under point-wise operations with norm $\|f\|_{\mathbb{A}}$.

The Commutative Banach Algebra $\mathbb{D}$

1. We associate to each of the operators A_j, $j = 1, \dots, n$, the formal object $\tilde{A}_j$, $j = 1, \dots, n$, by discarding all operator properties of A_j other than its operator norm $\|A_j\|_{\mathscr{L}(X)}$ and assuming that $\tilde{A}_i \tilde{A}_j = \tilde{A}_j \tilde{A}_i$ for any $1 \le i, j \le n$. In order to define $\mathbb{D}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$ we replace the indeterminates $z_1, \dots, z_n$ in functions $f(z_1, \dots, z_n) \in \mathbb{A}$ by $\tilde{A}_1, \dots, \tilde{A}_n$, respectively. That is, we take $\mathbb{D}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$ to be the collection of all expressions

$$f\left(\tilde{A}_1, \dots, \tilde{A}_n\right) = \sum_{m_1, \dots, m_n = 0}^{\infty} a_{m_1, \dots, m_n} \left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n}$$

for which

$$\|f\|_{\mathbb{D}(\tilde{A}_1, \dots, \tilde{A}_n)} = \|f\|_{\mathbb{D}} = \sum_{m_1, \dots, m_n = 0}^{\infty} |a_{m_1, \dots, m_n}| \, \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty.$$

Some Remarks about $\mathbb{A}$ and $\mathbb{D}$

- $\mathbb{D}$ is clearly a commutative Banach algebra under point-wise operations with norm $\|\cdot\|_{\mathbb{D}}$.
- $\mathbb{A}$ and $\mathbb{D}$ can be identified.
- We refer to $\mathbb{D}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ as the *disentangling algebra*.
- It is in the commutative algebra $\mathbb{D}$ that we will carry out the disentangling calculations for a given function. Once these calculations are done, we map the result to the corresponding operator in the noncommutative setting of $\mathscr{L}(X)$.

The Disentangling Map I

1. Let $A_1, \ldots, A_n \in \mathscr{L}(X)$.
2. Associate to each A_i a continuous Borel probability measure μ_i on $[0, T]$.
3. We need ordered sets of time indices. So, for $\pi \in S_m$, the set of permutations of $\{1, \ldots, m\}$, let

$$\Delta_m(\pi) = \{(s_1, \ldots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\}.$$

The Disentangling Map II

1. For $j = 1, \ldots, n$ and all $s \in [0, T]$, we let $\tilde{A}_j(s) \equiv \tilde{A}_j$.
2. For nonnegative integers $m_1, \ldots, m_n$ and $m = m_1 + \cdots + m_n$ define

$$\tilde{C}_i(s) = \begin{cases} \tilde{A}_1(s) & \text{if } i \in \{1, \ldots, m_1\}, \\ \tilde{A}_2(s) & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\ \quad \vdots & \\ \tilde{A}_n(s) & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}. \end{cases}$$

The Disentangling Map III

Critical for the definition of the disentangling map is:

Theorem

$$P^{m_1,\ldots,m_n}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right) = \tilde{A}_1^{m_1}\cdots\tilde{A}_n^{m_n} =$$

$$\sum_{\pi\in S_m}\int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)})\cdots\tilde{C}_{\pi(1)}(s_{\pi(1)})\cdot$$

$$(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n})(ds_1,\ldots,ds_m)$$

- This assertion is proved by writing $\tilde{A}_1^{m_1}\cdots\tilde{A}_n^{m_n} = \left\{\int_0^T \tilde{A}_1(s)\,\mu_1(ds)\right\}^{m_1}\cdots\left\{\int_0^T \tilde{A}_n(s)\,\mu_n(ds)\right\}^{m_n}$

The Disentangling Map IV

Definition

$$\mathscr{T}_{\mu_1,\ldots,\mu_n}\left(P^{m_1,\ldots,m_n}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)\right) :=$$

$$\sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \cdot$$

$$(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$

The Disentangling Map V

1. Let $f \in \mathbb{D}$ be given by $f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}$.

Definition

$$\mathscr{T}_{\mu_1,\ldots,\mu_n} f\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) =$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \mathscr{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$$

Facts about $\mathscr{T}_{\mu_1,\ldots,\mu_n}$

1. The series $\sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \mathscr{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1,\ldots,\tilde{A}_n\right)$ converges in operator norm.
2. The disentangling map is a linear norm one (in the time independent setting) contraction from $\mathbb{D}$ into $\mathscr{L}(X)$.
3. In the time-dependent setting where we have operator-valued functions $A_i : [0,T] \to \mathscr{L}(X)$, the definition of $\mathscr{T}_{\mu_1,\ldots,\mu_n}$ is the same; the definition of the algebras $\mathbb{A}$ and $\mathbb{D}$ explicitly involve the time-ordering measures and the disentangling map is no longer necessarily of norm one.
4. All of the above can be found in the 2001 paper [JJ1] (for the time independent setting considered here). See also [JLNBook].

Outline

A Symmetry Result

(Proposition 2.2 from the 2001 paper [JJ2].)

Theorem

Let $f \in \mathbb{A}(\|A_1\|, \ldots, \|A_n\|)$ *and suppose that there is some subsequence* $\{i_1, \ldots, i_\ell\}$ *from* $\{1, \ldots, n\}$ *such that the function* f *is symmetric in the variables* $\{z_{i_1}, \ldots, z_{i_\ell}\}$. *Then* f *belongs to* $\mathbb{A}(r_1, \ldots, r_n)$, *where* $r_1 = \|A_i\|$, $i \notin \{i_1, \ldots, i_\ell\}$, *and* $r_i = \max\{\|A_{i_1}\|, \ldots, \|A_{i_\ell}\|\}$, *if* $i \in \{i_1, \ldots, i_\ell\}$. *Further, any permutation of the operators* $A_{i_1}, \ldots, A_{i_\ell}$, *accompanied by the same permutation of the associated measures* $\mu_{i_1}, \ldots, \mu_{i_\ell}$, *leaves* $f_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$ *unchanged. Finally, if we have* $\mu_{i_1} = \cdots = \mu_{i_\ell}$, *then the function* $f_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n)$ *of the operators* $A_1, \ldots, A_n$ *is a symmetric function of* $A_{i_1}, \ldots, A_{i_\ell}$.

Effect of Ordered Supports on Disentangling

(Corollary 4.3 from [JJ2])

- Recall that the support $S(\mu)$ of the measure μ on $\mathscr{B}([0,T])$ is the complement of the set of all points x possessing an open neighborhood $U \in \mathscr{B}([0,T])$ such that $\mu(U) = 0$.

Theorem

Let $A_1, \dots, A_n \in \mathscr{L}(X)$. Associate to each A_i a continuous Borel probability measure μ_i on $[0,T]$. Let $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and suppose the supports of the measures $\mu_{i_1}, \dots, \mu_{i_k}$ are ordered as $S(\mu_{i_1}) \leq \cdots \leq S(\mu_{i_k})$. Let $\{j_1, \dots, j_{n-k}\} := \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$. Finally, assume that $S(\mu_{i_k}) \leq S(\mu_{j_p})$ for $p = 1, \dots, n-k$. Then $P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}(A_1, \dots, A_n) = P^{m_{j_1},\dots,m_{j_{n-k}}}_{\mu_{j_1},\dots,\mu_{j_{n-k}}}(A_{j_1}, \dots, A_{j_{n-k}})\, A_{i_k}^{m_{i_k}} \cdots A_{i_1}^{m_{i_1}}$.

Ordered Supports and Tensor Products

(Prop. 4.5 from [JJ2])

Theorem

Let $A_1, \ldots, A_n \in \mathscr{L}(X)$. Associate to each A_i a continuous Borel probability measure μ_i on $[0, T]$. Let $\{i_1, \ldots, i_n\}$ be any permutation of $\{1, \ldots, n\}$ and suppose that $S(\mu_{i_1}) \leq \cdots \leq S(\mu_{i_n})$. Further suppose that $g_i \in \mathbb{A}(\|A_i\|)$ be a function of one variable. Then

$$\mathscr{T}_{\mu_1,\ldots,\mu_n}(g_1 \otimes \cdots \otimes g_n)(\tilde{A}_1, \ldots, \tilde{A}_n) = g_{i_n}(A_{i_n}) \cdots g_{i_1}(A_{i_1}).$$

Disentangling Using One Measure

(Lemma 5.4 from [JJ2].)

Theorem

Let $A_1, \dots, A_n \in \mathscr{L}(X)$. Let μ be a continuous probability measure on $\mathscr{B}([0,T])$. For all scalars $\xi_1, \dots, \xi_n$ and all nonnegative integers m, we have

$$\mathscr{T}_{\mu,\dots,\mu}\left(\sum_{j=1}^{n} \xi_j \tilde{A}_j\right)^m = \left(\sum_{j=1}^{n} \xi_j A_j\right)^m .$$

- The lemma stated just above enables one to obtain the "Dyson series" expansion for $e^{(A+B)t}$ with respect to a continuous probability measure μ.

Iterative Disentangling

(Theorem 2.4 from the 2006 paper [JJK].)

Theorem

Let $A_1,\ldots,A_n \in \mathscr{L}(X)$*. Associate to each* A_i *a continuous Borel probability measure* μ_i *on* $[0,T]$*. Let* $a_j, b_j, j=1,\ldots,d$ *be real numbers such that* $0 \le a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_d \le b_d \le T$*. Suppose that* μ_i*,* $i \in \{i_{j-1}+1,\ldots,i_j\} =: I_j$ *have supports contained within* $[a_j,b_j]$ *for* $j=1,\ldots,d$*. Let* ν_j*,* $j=1,\ldots,d$ *be any continuous probability measures having supports contained within* $[a_j,b_j]$*. Given nonnegative integers* $m_1,\ldots,m_n$*, let* $K_j = P^{m_{i_{j-1}+1},\ldots,m_{i_j}}_{\mu_{i_{j-1}+1},\ldots,\mu_{i_j}}(A_{i_{j-1}+1},\ldots,A_{i_j})$*. Then* $P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)$ $= P^{1,\ldots,1;m_j,j\in I_0}_{\nu_1,\ldots,\nu_d;\mu_j,j\in I_0}(K_1,\ldots,K_d;A_j,j\in I_0)$ *where* $I_0 := \{1,\ldots,n\}\setminus(I_1\cup\cdots\cup I_d)$*.*

Effect of Commutativity

(From [JJ1].)

Theorem

Let $A_1, \dots, A_n \in \mathscr{L}(X)$. Associate to each A_i a continuous Borel probability measure μ_i on $[0, T]$. Suppose that $A_1 A_j = A_j A_1$ for $j = 2, \dots, n$. Then

$$\begin{aligned} P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}(A_1,\dots,A_n) &= A_1^{m_1} P^{m_2,\dots,m_n}_{\mu_2,\dots,\mu_n}(A_2,\dots,A_n) \\ &= P^{m_2,\dots,m_n}_{\mu_2,\dots,\mu_n}(A_2,\dots,A_n) A_1^{m_1}. \end{aligned}$$

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Another Representation of the Disentangling Map

(From *Feynman's Operational Calculi: Using Cauchy's Integral Formula* by LN, in preparation)

- To obtain a representation of the disentangling of a given function $f(z_1,\ldots,z_n)$, it is necessary, because of the definition of the norm $\|\cdot\|_{\mathbb{D}}$, to construct an algebra $\mathbb{D}_\infty$ of functions analytic on any polydisk containing the closed polydisk $\{(z_1,\ldots,z_n) : |z_j| \le \|A_j\|_{\mathscr{L}(X)}, j = 1,\ldots,n\}$.

Theorem

For $f \in \mathbb{D}_\infty$ we have

$$\mathscr{T}_{\mu_1,\ldots,\mu_n} f = (2\pi i)^{-n} \int_{|\xi_1|=r_{1,k_0}} \cdots \int_{|\xi_n|=r_{n,k_0}} f(\xi_1,\ldots,\xi_n) \mathscr{T}^{(k_0)}_{\mu_1,\ldots,\mu_n}\left(\left(\xi_1 - \tilde{A}_1\right)^{-1} \cdots \left(\xi_n - \tilde{A}_n\right)^{-1}\right) d\xi_1 \cdots d\xi_n$$

Spectral Theory for Feynman's Operational Calculus - I

The following comes from the 2007 paper [JeJoNiSpectral].

Definition

Let $A_1, \ldots, A_n \in \mathscr{L}(X)$. Let $\mu = (\mu_1, \ldots, \mu_n)$ be an n-tuple of continuous probability measures on $\mathscr{B}([0,1])$. If there are constants $C, r, s \geq 0$ such that

$$\left\| \mathscr{T}_{\mu_1,\ldots,\mu_n}\left(e^{i\langle\zeta,\tilde{\mathbf{A}}\rangle}\right)\right\|_{\mathscr{L}(X)} \leq C\,(1+|\zeta|)^s\, e^{r|\Im\zeta|}$$

for all $\zeta \in \mathbb{C}^n$, then the n-tuple of operators is said to be of Paley-Wiener type (s, r, μ).

- If this holds, there is a unique $\mathscr{L}(X)$-valued distribution $\mathscr{F}_{\mu,\mathbf{A}} \in \mathscr{L}(C^\infty(\mathbb{R}^n), \mathscr{L}(X))$ such that $\mathscr{F}_{\mu,\mathbf{A}}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathscr{T}_{\mu_1,\ldots,\mu_n}\left(e^{i\langle\zeta,\tilde{\mathbf{A}}\rangle}\right)\hat{f}(\xi)d\xi$ for every $f \in \mathscr{S}(\mathbb{R}^n)$.

Spectral Theory for Feynman's Operational Calculus - II

- The distribution $\mathscr{F}_{\mu,\mathbf{A}}$ obtained above is compactly supported and so of finite order. Its support $\gamma_\mu(\mathbf{A})$ is defined as the μ-joint spectum of the n-tuple $\mathbf{A}$.

Theorem

An n-tuple $\mathbf{A} = (A_1, \dots, A_n)$ of bounded self-adjoint operators on a Hilbert space $\mathscr{H}$ is of Paley-Wiener type $(0, r, \mu)$ with $r = \left(\|A_1\|^2 + \cdots + \|A_n\|^2\right)^{1/2}$, for any n-tuple $\mu = (\mu_1, \dots, \mu_n)$ of continuous probability measures on $\mathscr{B}([0,1])$.

An Integral Equation for Feynman's Operational Calculus

Using an evolution equation that appears in the book *[JLBook]* (and in [DJL] - '97), an integral equation for a "reduced" version of the disentangling can be found. A couple of preliminaries are needed.

1. Let $f(z_0, z_1, \ldots, z_n) = e^{z_0} g(z_1, \ldots, z_n)$ where g is an element of the appropriate disentangling algebra.
2. Let $\Phi^j(z_0, z_1, \ldots, z_n) = \frac{e^{z_0}}{z_j}(g(z_1, \ldots, z_n) - g(z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_n))$, for $j = 1, \ldots, n$.

The Integral Equation

With f and Φ^j given above, we can state the following:

Theorem

$$f^t_{(R);l;\mu_1,\ldots,\mu_n}(-\alpha, A_1(\cdot),\ldots,A_n(\cdot)) = g(0,\ldots,0)e^{-t\alpha}+$$
$$\sum_{j=1}^{n}\int_{[0,t]} e^{-(t-s)\alpha}A_j(s)\Phi^{j,s}_{(R);l;\mu_1,\ldots,\mu_n}(-\alpha, A_1(\cdot),\ldots,A_n(\cdot))\,\mu_j(ds)$$

- This equation appears in the 2008 paper [NiIntEqn].

Example

- Suppose we take A_1 to be multiplication by $V : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ on $L^2(\mathbb{R}^d)$ where $R_V := \int_{[0,T]} \|V(s,\cdot)\|_\infty ds < \infty$ (so Lebesgue measure is associated to this operator), $\alpha = H_0 = \frac{1}{2}\Delta$, and $f(z_0,z_1) = e^{z_0}(z_1 g_1(z_1) + g_1(0))$ where $g_1(z_1)$ is analytic on the disk $D(0,R_V)$ and continuous on its boundary. After some calculation we obtain, for any $\psi \in L^2(\mathbb{R}^d)$,

$$
f^t_{(R);l;l}(-H_0, V)\psi(x) = g_1(0)e^{-tH_0}\psi(x) +
$$

$$
\int_0^t \int_{C_0^t} \left(\sum_{m=0}^{\infty} \frac{a_m}{m!} \left[\int_0^s V(y(u)+x,u)\,du \right]^m \right) V(y(s)+x,s) \cdot
$$

$$
\psi(y(t)+x)\, m(dy)\, ds
$$

Example Continued

- If we take, above, $g_1(z_1) = \frac{1}{1-z_1}$ and $V : \mathbb{R}^d \to \mathbb{R}$ with $\|V\|_\infty < 1$, then

$$f^t_{(R);l;l}(-H_0, V)\psi(x) = e^{-t(H_0+V)}\psi(x)$$

  for $\psi \in L^2(\mathbb{R}^d)$. This follows from the formula on the previous slide and the Feynman-Kac formula.
- We can also use the g_1 and V just above to derive a connection between the disentangling $f^t_{(R);l,l}(-H_0, V)$ and the analytic in time operator-valued Feynman integral and the analytic (in mass) operator-valued Feynman integral.

Stability of the Calculi With Respect to the Time-Ordering Measures ([NiVectorFnStab] - 2007)

- Suppose that $\mathscr{H}$ is a separable Hilbert space. Let S be a metric space and let μ be a Borel probability measure on S and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel probability measures on S such that $\int_S f\,d\mu_n \to \int_S f\,d\mu$ for all bounded, continuous, real-valued f on S. (This says that $\mu_n \rightharpoonup \mu$; i.e. μ_n converges weakly to μ.)
- Suppose that we are given a continuous function $f : S \to \mathscr{H}$ with $\sup_{s\in S} \|f(s)\|_{\mathscr{H}} < \infty$.

Theorem

Under the assumptions above, $\lim_{n\to\infty} \int_S f\,d\mu_n = \int_S f\,d\mu$ *in norm on* $\mathscr{H}$.

- This theorem is used to prove...

The Stability Theorem ([NiVectorFnStab] - 2007)

Theorem

For $i=1,\ldots,n$ let $A_i : [0,T] \to \mathscr{L}(\mathscr{H})$, $\mathscr{H}$ a separable Hilbert space, be continuous. Associate to each $A_i(\cdot)$ a continuous Borel probability measure μ_i on $[0,T]$. For each $i=1,\ldots,n$, let $\{\mu_{ik}\}_{k=1}^{\infty}$ be a sequence of continuous Borel probability measures on $[0,T]$ such that $\mu_{ik} \rightharpoonup \mu$ as $k \to \infty$. Construct the direct sum Banach algebra $\mathscr{U}_D := \sum_{k \in \mathbb{N} \cup \{0\}} \bigoplus \mathbb{D}\left(\left(\tilde{A}_1(\cdot), \mu_{1,k}\right), \ldots, \left(\tilde{A}_n(\cdot), \mu_{n,k}\right)\right)$ where for $k=0$ the summand is $\mathbb{D}\left(\left(\tilde{A}_1(\cdot), \mu_1\right), \ldots, \left(\tilde{A}_n(\cdot), \mu_n\right)\right)$. Then, for any $\theta_f := (f,f,f,\ldots) \in \mathscr{U}_D$ and any $\phi \in \mathscr{H}$, we have

$$\lim_{k \to \infty} \left\| \mathscr{T}^{T}_{\mu_{1k},\ldots,\mu_{nk}}(\pi_k(\theta_f))\phi - \mathscr{T}^{T}_{\mu_1,\ldots,\mu_n}(\pi_0(\theta(f))\phi \right\|_{\mathscr{H}} = 0$$

where π_k is the canonical projection of $\mathscr{U}_D$ onto the disentangling algebra indexed by the measures $\mu_{1k},\ldots,\mu_{nk}$.

Derivational Derivatives

In their 2009 paper [JoKimDerDer] Johnson and Kim introduce derivational derivatives. They prove (Theorem 3.1):

Theorem

For each $f \in \mathbb{D}$ and for an arbitrary derivation D on $\mathscr{L}(X)$, we have

$$D\left[f_{\mu_1,\ldots,\mu_n}(A_1,\ldots,A_n)\right] = \sum_{j=1}^{n} (F_j)_{\mu_1,\ldots,\mu_j,\mu_j,\mu_{j+1},\ldots,\mu_n}(A_1,\ldots,A_j,D(A_j),A_{j+1},\ldots,A_n)$$

where $F_j(x_1,\ldots,x_j,y,x_{j+1},\ldots,x_n) = \frac{\partial}{\partial x_j} f(x_1,\ldots,x_n)y$.

- This theorem is first proved, via a somewhat tedious calculation, for D an inner derivation. It is then shown that the theorem is, in fact, true for an arbitrary derivation.

Taylor's Theorem with Remainder

Further on in their 2009 paper [JoKimDerDer], a version of Taylor's Theorem is obtained (Theorem 4.4).

Theorem

Let A, C be nonzero operators in $\mathscr{L}(X)$ and let μ_1, μ_2 be continuous probability measures on the Borel class $\mathscr{B}([0,T])$. for any positive integer N we have

$$f(C) - f(A) = \sum_{n=1}^{N} \frac{1}{n!} \mathscr{T}_{\mu_1,\mu_2} h_n(\tilde{A}, \tilde{C}) + R_N,$$

where $h_n(x,y) = f^{(n)}(x)(y-x)^n$ for $n = 1, 2, \ldots, N$ and $R_N = \frac{1}{(N+1)!}\mathscr{T}_{\mu_1,\mu_2}\left\{ f^{(N+1)}(\tilde{A} + t(\tilde{C} - \tilde{A}))(\tilde{C} - \tilde{A})^{N+1} \right\}$ for some $0 < t < 1$.

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