INVERSE FUNCTION THEOREM

I use $df_x$ for the linear transformation that is the differential of $f$ at $x$.

**Definition 1.** Suppose $S \subseteq \mathbb{R}^n$ is open, $a \in S$, and $f : S \to \mathbb{R}^n$ is a function. We say $f$ is **locally invertible around $a$** if there is an open set $A \subseteq S$ containing $a$ so that $f(A)$ is open and there is a function $g : f(A) \to A$ so that, for all $x \in A$ and $y \in f(A)$,

$$g(f(x)) = x, \quad f(g(y)) = y.$$  

Clearly, it suffices to have $f(A)$ open and $f$ one-to-one on the open set $A$. It is important to note how $f^{-1}$ depends on the choice of $A$. If $B$ another open set and $h : f(B) \to B$ is an inverse for $f$ on $B$, then on $A \cap B$, $h$ and $g$ agree. So changing the set $A$ may change the domain of $f^{-1}$ but not the value of $f^{-1}(x)$ for any point $x$.

**Definition 2.** If $S \subseteq \mathbb{R}^n$ is open, then $g : S \to \mathbb{R}^m$ is **Lipschitz** if there is a constant $K$ so that

$$\|g(w) - g(y)\| \leq K\|w - y\|.$$  

We will need the following result:

**Proposition 3.** Linear transformations are Lipschitz. That is, for a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, there is $M > 0$ so that, for all $x, y \in \mathbb{R}^n$,

$$\|Lx - Ly\| \leq M\|x - y\|.$$  

We also need the following result:

**Proposition 4.** Let $S \subseteq \mathbb{R}^n$ is open. If function $f : S \to \mathbb{R}$ is continuous and $T \subseteq S$ is a closed and bounded set, then $f$ attains its maximum and minimum on $T$. That is, there is $t_0, t_1 \in T$ so that, for all $t \in T$,

$$f(t_0) \leq f(t) \leq f(t_1).$$  

Note that $f$ does not need to have an inverse function for $f^{-1}(V)$ to make sense.

**Theorem 5** (Local Invertibility). Let $S \subseteq \mathbb{R}^n$ is open, $a \in S$, and $f : S \to \mathbb{R}^n$ is $C^1$. If $df_a$ is invertible, then $f$ is locally invertible around $a$ and $f^{-1}$ is Lipschitz.

**Lemma 6.** With the same hypotheses as the theorem, there are $\epsilon, c > 0$ so that, for all $x, z \in B_\epsilon(a)$,

$$\|f(x) - f(z)\| \geq c\|x - z\|.$$  

and, for all $x \in B_\epsilon(a)$, $df_x$ is invertible.

**Proof of Local Invertibility Theorem.** Using the lemma, observe that for $x, z \in B_\epsilon(a)$ with $x \neq z$,

$$\|f(x) - f(z)\| \geq c\|x - z\| > 0$$  

and so $f(x) \neq f(z)$, i.e. $f$ is one-to-one on $B_\epsilon(a)$. Thus, there is a function $f^{-1} : f(B_\epsilon(a)) \to B_\epsilon(a)$. Moreover, for $w, y \in f(B_\epsilon(a))$, there are $x, z \in B_\epsilon(a)$ with $w = f(x)$ and $y = f(z)$. Using (7),

$$\|w - y\| \geq c\|f^{-1}(w) - f^{-1}(y)\|.$$
This shows \( f^{-1} \) is Lipschitz (with constant \( 1/c \)) and so is continuous.

To see that \( f(B_c(a)) \) is open, fix \( v \) in this set. There is \( x \in B_c(a) \) with \( f(x) = v \). Choose \( s > 0 \) so that \( B_s(x) \) is contained in \( B_c(a) \). Then \( S = \{ y : \|y - x\| = s \} \), the boundary of \( B_s(x) \), is a closed and bounded set. Since \( f \) is continuous, the image, \( f(S) \), is also closed and bounded. By the proposition, there is \( y_0 \in S \) so that the function \( z \mapsto \|f(z) - v\| \) attains its minimum. That is, for all \( y \in S \),

\[
\|f(y) - v\| \geq \|f(y_0) - v\|.
\]

As \( f \) is one-to-one, \( v \) is not in \( f(S) \); so \( d = \|f(y_0) - v\| > 0 \).

We shall show that \( B_{d/2}(v) \) is contained in \( f(B_c(a)) \). Let \( u \in B_{d/2}(v) \) and define a function on \( B_s(x) \) by

\[
g(y) = \|f(y) - u\|^2 = (f(y) - u) \cdot (f(y) - u).
\]

Observe that \( g \) is \( C^1 \) because \( f \) is and by previous work

\[
Dg_y(h) = 2(Df_y)(h) \cdot (f(y) - u).
\]

Since \( B_s(x) \) is a closed and bounded set and \( g \) is continuous, the proposition guarantees that \( g \) attains its minimum value. Observe that at every point of \( S \),

\[
g(y) = \|f(y) - u\|^2 \geq (\|f(y) - v\| - \|v - u\|)^2 \geq \left( d - \frac{d^2}{2} \right) = \frac{d^2}{4},
\]

while

\[
g(x) = \|v - u\|^2 < \frac{d^2}{4}.
\]

Hence the minimum of \( g \) occurs at some interior point \( y_0 \). So by previous work, \( Dg(y_0) = 0 \). But \( df(y_0) \) is invertible by the lemma, so \( f(y_0) - u = 0 \); that is, \( f(y_0) = u \). Therefore \( f(B_c(a)) \) is open.

**Proof of Lemma.** Let \( T = (df_a)^{-1} \). By the proposition above, there is \( M > 0 \) so that

\[
\|Tu - Tv\| \leq M\|u - v\|.
\]

Letting \( u = T^{-1}(x - a) \) and \( v = T^{-1}(y - a) \), (so \( u = df_a(x - a) \)), we have

\[
\|df_a(x - a) - df_a(y - a)\| \leq \frac{1}{M} \|x - y\|.
\]

Define \( E : S \to \mathbb{R}^n \) by \( E(x) = f(x) - f(a) - df_a(x - a) \). Since \( f \) is \( C^1 \) and linear transformations are infinitely differentiable, \( E \) is \( C^1 \). Notice that

\[
dE_a(h) = df_a(h) - df_a(h) = 0.
\]

In particular, if \( E = (E_1, \ldots, E_n) \), then by the continuity of \( d(E_i)_a \) there is some \( \varepsilon > 0 \) so that

\[
\|d(E_i)z\| \leq \frac{1}{2M\sqrt{n}},
\]

for \( i = 1, \ldots, n \) and all \( z \in B_c(a) \).

Suppose that \( x, z \in B_c(a) \). Then, for each \( i \), by Taylor’s Theorem with linear remainder term, there is \( c_i \in L[x, z] \subset B_c(a) \) so that

\[
|E_i(x) - E_i(z)| = |d(E_i)_{c_i}(x - z)| \leq \frac{1}{2M\sqrt{n}} \|x - z\|.
\]
Using the inverse function identities and moving

\[ \text{if there was} \]

\[ \text{have that} \]

\[ \text{c} \]

The proves (7) with

\[ \text{must be invertible}. \]

Thus, \( \| E(x) - E(z) \| \leq \| x - z \| / (2M) \).

As \( f(x) - f(z) = E(x) - E(z) - (d f_a(x - a) - d f_a(z - a)) \),

\[ \| f(x) - f(z) \| \geq \| d f_a(x - a) - d f_a(z - a) \| - \| E(x) - E(z) \| \]

\[ \geq \frac{1}{M} \| x - z \| - \frac{1}{2M} \| x - z \| = \frac{1}{2M} \| x - z \|. \]

The proves (7) with \( c = 1/(2M) \).

Finally, to see that \( d f_a \) in invertible for each \( x \in B_c(a) \), observe that

\[ d E_x(z - x) = d f_a(z - x) - d f_a(z - x). \]

If there was \( z \) so that \( d f_a(z - x) = 0 \), then \( d E_x(z - x) = -d f_a(z - x) \). On the other hand, we have that

\[ \| d f_a(z - x) \| \geq \frac{1}{M} \| z - x \|. \quad \| d E_x(z - x) \| \leq \frac{1}{2M} \| z - x \|. \]

This contradiction shows that \( d f_a \) must be invertible.

Recall that we proved that a function \( g \) is differentiable at \( c \) if and only if there is a linear transformation \( L \) and a function \( \epsilon \) so that \( \lim_{x \to c} \epsilon(x) = 0 \) and

\[ g(x) = g(c) + L(x - c) + \epsilon(x) \| x - c \|. \]

In this case, \( L \) is \( d g_e \).

**Theorem 8** (Inverse Function Theorem). Let \( S \subseteq \mathbb{R}^n \) be open, \( a \in S \), and \( f : S \to \mathbb{R}^n \) is \( C^1 \). If \( d f_a \) is invertible, then \( f^{-1} \) is differentiable at \( b = f(a) \) and

\[ d(f^{-1})_b = (d f_{f^{-1}(b)})^{-1}. \]

**Proof.** Since \( f \) is differentiable at \( a \), there is a function \( \epsilon : S \to \mathbb{R}^n \) with \( \lim_{x \to a} \epsilon(x) = 0 \) and

\[ f(x) = f(a) + d f_a(x - a) + \epsilon(x) \| x - a \|. \]

Since \( f \) is locally invertible around \( a \), there is some open set \( A \) containing \( a \) on which \( f \) is one-to-one and \( f^{-1} \) is Lipschitz on the open set \( f(A) \).

For \( x \in A \), there is \( y \in f(A) \) with \( x = f^{-1}(y) \). Using this and \( a = f^{-1}(b) \), we have

\[ f(f^{-1}(y)) = f(f^{-1}(b)) + d f(a)(f^{-1}(y) - f^{-1}(b)) + \epsilon(f^{-1}(y)) \| f^{-1}(y) - f^{-1}(b) \|. \]

Using the inverse function identities and moving \( b \) over, we have

\[ y - b = d f_a(f^{-1}(y) - f^{-1}(b)) + \epsilon(f^{-1}(y)) \| f^{-1}(y) - f^{-1}(b) \|. \]
Applying \((df_a)^{-1}\) to this equation and using the linearity of \((df_a)^{-1}\), we have
\[
(df_a)^{-1}(y - b) = f^{-1}(y) - f^{-1}(b) + (df_a)^{-1}(\epsilon(f^{-1}(y)))\|f^{-1}(y) - f^{-1}(b)\|.
\]
Then we can rearrange the previous equation to obtain
\[
f^{-1}(y) = f^{-1}(b) + (df_a)^{-1}(y - b) + \eta(y)\|y - b\|.
\]
if we define a new function \(\eta\) on \(f(A)\) by letting \(\eta(b) = 0\) and otherwise
\[
\eta(y) = \frac{-(df_a)^{-1}(\epsilon(f^{-1}(y)))\|f^{-1}(y) - f^{-1}(b)\|}{\|y - b\|}.
\]
To show that \(f^{-1}\) is differentiable at \(b\) and \(d(f^{-1})_b\) is \((df_a)^{-1}\), it suffices to show that
\[
\lim_{y \to b} \eta(y) = 0.
\]
As \(f^{-1}\) is Lipschitz, there is a constant \(K > 0\) so that
\[
\frac{\|f^{-1}(y) - f^{-1}(b)\|}{\|y - b\|} \leq K
\]
for all \(y \in f(A)\). So it suffices to prove that
\[
\lim_{y \to b} -(df_a)^{-1}(\epsilon(f^{-1}(y))) = 0.
\]
Now, as \(y \to b\), \(f^{-1}(y) \to f^{-1}(b) = a\). By our choice of the function \(\epsilon\), as \(f^{-1}(y) \to a\), \(\epsilon(f^{-1}(y)) \to 0\). Since the linear transformation \((df_a)^{-1}\) is continuous, we have the claimed limit.

This concludes the proof.

**Corollary 9.** \(f^{-1}\) is \(C^1\) on its domain.

This is very rough. Notice first that since \(f^{-1}\) is uniquely defined on its domain, call it \(A\), \(f\) is locally invertible at each point of \(A\). By the lemma, we may assume \(df_a\) is invertible for each \(a \in A\). By the inverse function theorem, we have that \(d(f^{-1})_b = (df_{f^{-1}(b)})^{-1}\) for each \(b \in f(A)\).

To see that this function is continuous, observe first that \(f^{-1}\) is continuous; second, that the map \(x \mapsto df_x\) is continuous; and third, that matrix inversion is continuous. As a composition of three continuous operations, \(d(f^{-1})_b\) is a continuous function of \(b\).