Due: Dec 3rd

1. Consider the equations
   \[ xu^2 + yv^2 + xy = 9, \]
   \[ xv^2 + yu^2 - xy = 7. \]

   Find conditions on \((x_0, y_0, u_0, v_0) \in \mathbb{R}^4\) so that there are real-valued \(C^1\) functions \(u(x, y)\) and \(v(x, y)\) that solve these equations for \(u\) and \(v\) in terms of \(x\) and \(y\) around this point. Prove that the solutions satisfy \(u^2 + v^2 = 16/(x+y)\).

2. In the Implicit Function Theorem, we take a function \(G: S \to \mathbb{R}^n\), where \(S \subset \mathbb{R}^{n+m}\) and separate the \(n + m\) invariables into \(y\), which we are solving for, and \(x\), which we are solving in terms of. The theorem then gives a condition in terms of this choice.

   Suppose you don’t care which variables are solved for or in terms of. Give a condition on the \(n \times (m+n)\) matrix of \(dG_c\), where \(c \in \mathbb{R}^{n+m}\) that allows you to solve for some (unspecified) choice of \(n\) of the variables in terms of the other \(m\). In terms of your condition, how do you choose the \(n\) variables to solve for?

   Prove that your answers are correct, of course.

   **HINT:** This is really more of a linear algebra exercise than an calculus one; your condition should be a familiar one from linear algebra.

3. For a \(C^1\) function \(G: \mathbb{R}^n \to \mathbb{R}\), let \(S = \{x \in \mathbb{R}^n : G(x) = 0\}\). If \(dG_a \neq 0\) for some \(a \in \mathbb{R}^n\), show that there is an open set \(N \subseteq \mathbb{R}^n\) so that \(S \cap N\) is the graph of a \(C^1\) function \(f\) from a suitable subset of \(\mathbb{R}^n\) into \(\mathbb{R}\).

4. Let \(0 = (0, 0, 0)\) and define \(T: \mathbb{R}^3\{0\} \to \mathbb{R}^3\{0\}\) by \(T(x) = x/\|x\|^2\), that is,
   \[
   T(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2}\right).
   \]

   (a) Show that \(T\) is globally invertible.

   (b) Find \(JT_x\). **HINT:** First use spherical coordinates.

   (c) Show that \(T\) maps \(z = -1\) onto \(\partial(B_{1/2}(0, 0, -1/2))\{0\}\).

   (d) (bonus question) Show that, for any plane, \(P\), not containing \(0\), \(T(P)\) is a sphere containing \(0\) whose tangent plane at \(0\) is parallel to \(P\).

   **HINT:** This is really a geometry problem. Start by considering the map \(I\) from \(\mathbb{R}^2\{0, 0\}\) to itself that sends \((x, y)\) to \((x/(x^2 + y^2), y/(x^2 + y^2))\) and show that a circle containing \((0, 0)\) is mapped to a line. Use this result to get the conclusion.