1. Let \( x : (a, b) \rightarrow \mathbb{R} \) be a regular, twice differentiable curve. Recall that \( T = \frac{x'}{\|x'\|} \). Define the curvature vector \( K \) to be the derivative of \( T \) with respect to arc length, \( s \).

(a) Show that \( K = \frac{T'}{\|x'\|} \) and \( \kappa = \|K\| \). HINT: \( \frac{dx}{ds} = \|x'\| \).

(b) For the curve \( x(t) = ti + (t^3/3)j \), verify the following
\[ T = i + t^2j \sqrt{1 + t^4}, \quad K = -\frac{2t(t^2 i - j)}{(1 + t^4)^{3/2}}, \quad \kappa = \frac{2|t|}{(1 + t^4)^{3/2}}. \]

2. Let \( x : (a, b) \rightarrow \mathbb{R}^3 \) be a regular, twice differentiable curve. Show that
\[ \kappa = \frac{\|x' \times x''\|}{\|x'\|^3}. \]
HINT: Write \( x' \) and \( x'' \) in terms of \( \frac{dx}{ds} \) and \( \frac{d^2x}{ds^2} \) and use the properties of the latter derivatives, such as \( \kappa = \|\frac{d^2x}{ds^2}\| \).

3. Do Problem 3.14 (page 89) in Edwards. That is, the following example illustrates the hazards of denoting functions by real variables. Let \( w = f(x, y, z) \) and \( z = g(x, y) \). Then
\[
\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial x},
\]
since \( \frac{\partial x}{\partial x} = 1 \) and \( \frac{\partial y}{\partial x} = 0 \). Hence \( \frac{\partial w}{\partial x} \frac{\partial z}{\partial x} = 0 \). But if \( w = x + y + z \) and \( z = x + y \), then \( \frac{\partial w}{\partial z} = \frac{\partial z}{\partial x} = 1 \), so we have \( 1 = 0 \). Where is the mistake?

4. Do Problems 6.80 and 6.81 in Schaum’s. We reword these problems slightly, to fit the discussion on page 80-81 of Edwards, which you might want to read.

Suppose \( u, v : \mathbb{R}^2 \rightarrow \mathbb{R} \) are functions with continuous second derivatives. Let \( U = u \circ T \) and \( V = v \circ T \), where \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is given by \( T(\rho, \phi) = (\rho \cos(\phi), \rho \sin(\phi)) \). That is, \( T \) is the transformation between polar and rectangular coordinates, given by \( x = \rho \cos \phi \) and \( y = \rho \sin \phi \).

(a) Show that, under this transformation, the equations \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) become
\[
\frac{\partial U}{\partial \rho} = \frac{1}{\rho} \frac{\partial V}{\partial \phi}, \quad \frac{\partial V}{\partial \rho} = -\frac{1}{\rho} \frac{\partial U}{\partial \phi}.
\]

(b) Show that Laplace’s equation, \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) becomes \( \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial \phi^2} = 0 \).