Due: Feb 18th

1. For each of the following sequences, find the limit and prove your answer is correct using the definition:

a) \( \lim_{n \to \infty} \frac{1}{n^{1/3}} \),

b) \( \lim_{n \to \infty} \frac{n}{n^2 + 1} \),

c) \( \lim_{n \to \infty} \frac{2n + 4}{5n + 2} \).

2. Suppose that the sequences \((s_n)\) and \((t_n)\) satisfy \( \lim_{n \to \infty} s_n = 0 \) and there is \( M > 0 \) so that for all \( n \in \mathbb{N} \), \( |t_n| \leq M \). Prove that \( \lim_{n \to \infty} s_n t_n = 0 \).

3. Let \((s_n)\) be a sequence of real numbers so that \( \lim_{n \to \infty} s_n = s \in \mathbb{R} \) and let \( a \in \mathbb{R} \). Prove the following:
   
   (a) If, for all but finitely many \( n \in \mathbb{N} \), \( s_n \geq a \), then \( s \geq a \).
   
   (b) If \( s > a \), then for all but finitely many \( n \in \mathbb{N} \), \( s_n \geq a \).
   
   (c) Give an example of a sequence \((s_n)\) and \( s \) as above, along with a number \( a \), so that \( s \geq a \) and there are infinitely many \( n \in \mathbb{N} \) with \( s_n < a \).

4. Suppose that there is \( N_0 \) so that for all \( n \geq N_0 \), \( s_n \leq t_n \).
   
   (a) Prove that if \( \lim_{n \to \infty} s_n = +\infty \) then \( \lim_{n \to \infty} t_n = +\infty \).
   
   (b) Prove that if \( \lim_{n \to \infty} t_n = -\infty \) then \( \lim_{n \to \infty} s_n = -\infty \).
   
   (c) Prove the if \( \lim_{n \to \infty} s_n = s \in \mathbb{R} \cup \{\pm \infty\} \) and \( \lim_{n \to \infty} t_n = t \in \mathbb{R} \cup \{\pm \infty\} \), then \( s \leq t \).

5. Show that if the sequence \((s_n)\) satisfies \( |s_n - s_{n+1}| < 2^{-n} \), then \((s_n)\) is Cauchy and so converges.

6. Extra Credit: Fix two real numbers \( a \) and \( b \). Define a sequence \((x_n)\) by \( x_1 = a \), \( x_2 = b \) and \( x_n = (x_{n-1} + x_{n-2})/2 \) for \( n \geq 2 \). Find \( \lim_{n \to \infty} x_n \).