Due: Feb 11th

1. If \( A = \{a + a^{-1} : a > 0\} \), then find \( \inf A \) and \( \min A \). Prove your answers (which may be that it doesn’t exist) are correct.

2. Suppose that \( S \) and \( T \) are bounded nonempty subsets of \( \mathbb{R} \).
   (a) If \( S \subseteq T \), prove that \( \inf T \leq \inf S \leq \sup S \leq \sup T \).
   (b) Prove that \( \sup(S \cup T) = \max\{\sup S, \sup T\} \). (Do not assume \( S \) is a subset of \( T \).)
   (c) Find a formula expressing \( \inf(S \cup T) \) in terms of \( \inf S \) and \( \inf T \). You do not need to prove it is correct.

3. Let \( S \) and \( T \) be nonempty subsets of \( \mathbb{R} \) with the following property: \( s \leq t \) for all \( s \in S \) and all \( t \in T \).
   (a) Observe that \( S \) is bounded above and \( T \) is bounded below.
   (b) Prove that \( \sup S \leq \inf T \).
   (c) Give an example of such sets \( S \) and \( T \) where \( S \cap T \) is nonempty.
   (d) Give an example of such sets \( S \) and \( T \) where \( S \cap T = \emptyset \) and \( \sup S = \inf T \).

4. Let \( \mathbb{I} \) be the set of all irrational numbers, i.e., all real numbers that are not rational numbers. Prove that if \( a, b \in \mathbb{R} \) and \( a < b \), then there is \( x \in \mathbb{I} \) with

\[
a < x < b.
\]

Hint: First show that \( \{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I} \). You may assume \( \sqrt{2} \in \mathbb{I} \).

5. Let \( A \) and \( B \) be nonempty bounded subsets of \( \mathbb{R} \). If \( S \) is the set of all sums \( a+b \) where \( a \in A \) and \( b \in B \), then prove the following:
   (a) \( \sup S = \sup A + \sup B \), and
   (b) \( \inf S = \inf A + \inf B \).

6. Extra Credit: If \( A \) is a nonempty subset of \( \mathbb{R} \), we call \( b \) an **almost upper bound** for \( A \) if there are only finitely many numbers \( a \in A \) with \( a \geq b \).
   (a) Find all almost upper bounds for each of \( \{n/(n+1) : n \in \mathbb{N}\} \) and \( \{1/n : n \in \mathbb{N}\} \).
   (b) If \( A \) is a bounded infinite set, prove that the set of all almost upper bounds is nonempty and bounded below.
   (c) Find an infinite subset of \( \mathbb{R} \) so that the set of all almost upper bounds is \( \mathbb{R} \).