Math 325-001 Assignment # 3 28 Jan 09

Due: Feb 4th

1. Recall that the definition of $\mathbb{Z}$ from class: letting $S = \{(a,b) : a,b \in \mathbb{N}\}$ and $(a,b) \sim (m,n)$ if and only if $m+b = a+n$, then $\mathbb{Z} = S/\sim$. We use $[(a,b)]$ for the equivalence class of (a,b) under $\sim$. We defined addition by $[(a,b)] + [(m,n)] = [(m+a, n+b)]$ and multiplication by
$$[(a,b)] \times [(m,n)] = [(ma+nb, na+mb)].$$
   (a) Show that $[(a,b)] + [(1,1)] = [(a,b)]$; hence we think of $[(1,1)]$ as the 'zero' in $\mathbb{Z}$.
   (b) Show that for each $[(a,b)]$, there is a unique $[(c,d)]$ with $[(a,b)] + [(c,d)] = [(1,1)]$. We denote $[(c,d)]$ by $-[(a,b)]$.
   (c) Show that multiplication is well defined.
   (d) Show that $[(a,b)] \times [(2,1)] = [(a,b)]$; think of $[(2,1)]$ as the 'multiplicative identity' in $\mathbb{Z}$.
   (e) Define a map $f : \mathbb{N} \to S/\sim$ so that $f(m) + f(n) = f(m+n)$ and $f(m) \times f(n) = f(m \times n)$ and show it satisfies these equations.
   (Note that the operations on the lefthand sides is the ones we defined above; on the righthand sides, they are the usual operations in $\mathbb{N}$.)
   (f) Extra Credit: Show that f is one-to-one but not onto.

2. Using only the field axioms and the first few results of Theorem 3.1, prove that, for all $a,b,c \in \mathbb{F}$, $\mathbb{F}$ a field, $ac = bc$ and $c \neq 0$ imply that $a = b$.

3. Using only the ordered field axioms, Theorem 3.1, and the first few results of Theorem 3.2, prove, for an ordered field $\mathbb{F}$,
   (a) $0 < 1$, and
   (b) For all $a,b \in \mathbb{F}$, $|b| \leq a$ if and only if $-a \leq b \leq a$.
   (c) For all $a,b \in \mathbb{F}$, $\big||a| - |b|\big| \leq |a-b|$.

   For the last two parts, you are allowed to use the definition of absolute value and the previous parts.

4. Prove, for $h \in \mathbb{R}$, $h > -1$ and $n \in \mathbb{N}$, that $(1+h)^n \geq 1 + nh$.