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1. The system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x}$ has solutions $\mathbf{x}_1(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}$ and $\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}$. Find the particular solution of the system that satisfies $\mathbf{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$.

   Solution. The general solution of the DE is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}.$$

   Using the initial conditions, we have

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \mathbf{x}(0) = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}$$

   and so we have the equations

$$3C_1 + C_2 = 8, \qquad 2C_1 + 3C_2 = 0.$$

   Taking 3 times the first equation minus the second gives $9C_1 - 2C_1 = 24$ and so $C_1 = 24/7$. Substituting this into the second equation gives $3C_2 = -48/7$ and so $C_2 = -16/7$. Thus, the particular solution is

$$\mathbf{x}(t) = \frac{24}{7} \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + \frac{-16}{7} C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}.$$

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2. Find the general solution of the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

   Solution. First, we find the eigenvalues, by solving

$$\begin{aligned} \det\left(\begin{bmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{bmatrix} \right) &= 0 \\ (1-\lambda)(5-\lambda) + 4 &= 0 \\ 9 - 6\lambda + \lambda^2 &= 0 \\ (\lambda - 3)^2 &= 0 \end{aligned}$$

   Thus, we have a repeated eigenvalue of $\lambda = 3, 3$.
   To find the eigenvectors, we look at

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This matrix is, essentially, the equation $v_1 + 2v_2 = 0$ twice, so $v_1 = -2v_2$. Letting $v_2 = 1$, we have $v_1 = -2$. This is the only linearly independent eigenvector, so we don't have enough eigenvectors. Notice that

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so we can pick $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

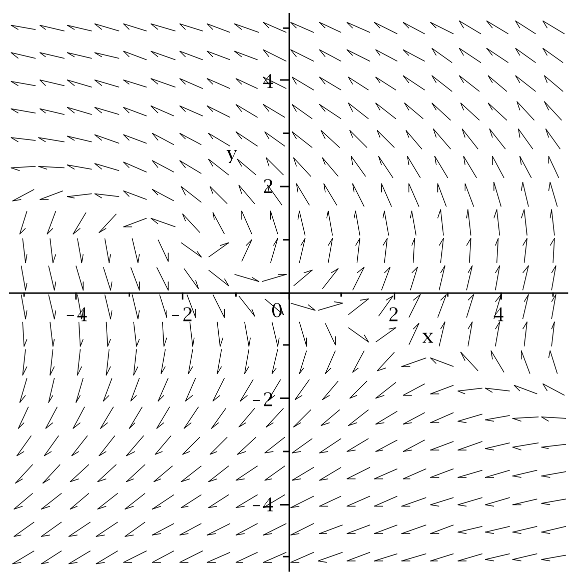
Thus, the general solution to the system is

$$\mathbf{x}(t) = C_1 \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{3t}.$$

. 10 3. Which of the following systems has the direction field given below? Give a (short) justification for your answer.

System 1 $\dfrac{dx}{dt} = 1 - y^2, \qquad \dfrac{dy}{dt} = x + 2y.$

System 2 $\dfrac{dx}{dt} = x - 2y, \qquad \dfrac{dy}{dt} = 4x - x^3.$



Solution. System 1 is the right system. It is not enough to find a property of the direction field that System 1 has. You must show that System 2 does not have this property.

There are many correct reasons possible, using nullclines, equilibrium solutions, or directions at points.

For example, System 2 has an equilibrium solution at (0,0) and the direction field does not, so System 2 cannot be correct.

System 2 has a y-nullcline (i.e., a curve where all the arrows are horizontal on the line $x = 0$. Since there are non-horizontal arrows on the line $x = 0$, System 2 cannot be correct.

At the point $(0, 1)$, System 2 has direction $(-2, 0)$ but the direction field is going straight up (i.e., something like $(0, 1)$), so System 2 cannot be correct.

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4. The system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 5 & -2 & -2 \\ 7 & -4 & -2 \\ 3 & 1 & -1 \end{bmatrix} \mathbf{x}$ has eigenvalues -2 and $1 \pm 2i$. Find the general solution of this system.

Solution. We need to find 3 linearly independent solutions, by finding eigenvectors. For the eigenvalue $\lambda = 2$, we have to solve the system

$$\begin{bmatrix} 7 & -2 & -2 \\ 7 & -2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us two equations, $7v_1 - 2v_2 - 2v_3 = 0$ and $3v_1 + v_2 + v_3 = 0$. Taking the first equation plus twice the second gives $13v_1 = 0$, so $v_1 = 0$. Then the second or third equation gives $v_2 + v_3 = 0$, so we can take $v_2 = 1$ and $v_3 = -1$.

For the eigenvalue $\lambda = 1 + 2i$, we have

$$\begin{bmatrix} 4-2i & -2 & -2 \\ 7 & -5-2i & -2 \\ 3 & 1 & -1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking the first row minus the second gives the equation $(-3-2i)v_1 + (3+2i)v_2 = 0$, and so we have $v_1 = v_2$. Let $v_1 = 1$, so $v_2 = 1$. Substituting this into the first equation gives $(4 - 2i) - 2 - 2v_3 = 0$ and so $v_3 = (2 - 2i)/2 = 1 - i$. Thus, the complex solution for this eigenvalue is

$$\begin{bmatrix} 1 \\ 1 \\ 1-i \end{bmatrix} e^{(1+2i)t}.$$

Now, we use Euler's formula, $e^{(1+2i)t} = e^t e^{2it} = e^t(\cos 2t + i \sin 2t)$, to find real and imaginary parts

$$\begin{bmatrix} 1 \\ 1 \\ 1-i \end{bmatrix} e^t(\cos 2t + i \sin 2t) = \begin{bmatrix} \cos 2t \\ \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} e^t + i \begin{bmatrix} \sin 2t \\ \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} e^t.$$

Thus, the general solution is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} \cos 2t \\ \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} e^{t} + C_3 \begin{bmatrix} \sin 2t \\ \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} e^{t}.$$

. 25 5. For the following system, graph all nullclines (including direction arrows on the nullclines) and find all equilibrium points. You need only consider the region $x \geq 0$ and $y \geq 0$.

$$\frac{dx}{dt} = x(3 - 2y - x), \quad \frac{dy}{dt} = y(2 - x - y).$$

Solution. In an effort to make solutions available quickly, I'm not drawing the graphics here. Feel free to ask me in class about this.

We find the nullclines by solving $\frac{dx}{dt} = 0$, i.e. $x(3 - 2y - x) = 0$, which gives $x = 0$ and $x + 2y = 3$, and by solving $\frac{dy}{dt} = 0$, i.e., $y(2 - x - y) = 0$, which gives $y = 0$ and $x + y = 2$.

There are four equilibrium solutions (where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both zero), namely $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(1, 1)$.

On the x-nullcline $x + 2y = 3$, the arrow is up between $(0, 3/2)$ and $(1, 1)$ and down between $(1, 1)$ and $(3, 0)$. On the y-nullcline $x + y = 2$, the arrow is left between $(0, 2)$ and $(1, 1)$ and right between $(1, 1)$ and $(2, 0)$.