Math 221-601 **Answers to Test #3** **3 Aug 07**

. 10 1. Give the form of the particular solution (the guess) for

$$y^{(3)} - y' = e^x + 4e^{-x} + \cos 2x.$$

You do not need to find the values of the constants in the guess.

Solution. The homogeneous DE is $(D^3 - D)[y] = 0$, so the roots of the characteristic equation is $-1, 0, 1$ and the general solution is $y = C_1 + C_2 e^x + C_3 e^{-x}$. The annhilator of the righthand side is $(D+1)(D-1)(D^2+4)$. Applying this to the original DE gives $(D+1)^2(D-1)^2(D^2+4)D[y]0$, which has solution

$$y = C_1 + C_2 e^x + C_3 e^{-x} + C_4 x e^x + C_5 x e^{-x} + C_6 \sin 2x + C_7 \cos 2x.$$

Thus, the guess is

$$y = C_4 x e^x + C_5 x e^{-x} + C_6 \sin 2x + C_7 \cos 2x.$$

. 15 2. Here are two lists of solutions to two different DEs. For each list, are the three functions linearly independent or linearly dependent?

(a) x, e^{2x}, e^x

(b) $x^2 - 1$, $x + 1$, $3x + 3$

Solution. For the first list, use the Wronskian

$$W(x, e^{2x}, e^x) = \begin{vmatrix} x & e^{2x} & e^x \\ 1 & 2e^{2x} & e^x \\ 0 & 4e^{4x} & e^x \end{vmatrix}.$$

We let $x = 0$, to obtain

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 4 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 3 \neq 0.$$

Since the Wronskian is not zero, the functions are linearly independent.

For the second list, we observe that

$$0(x^2 - 1) + (-3)(x + 1) + 1(3x + 3) = 0.$$

and since the constants are not all zero, by the definition, the three functions are linearly dependent.

. 15 3. Give the general solution of $y^{(4)} - y^{(3)} + 9y'' - 9y' = 0$.

Solution. The DE can be factored as follows:

$$\begin{aligned}(D^4 - D^3 + 9D^2 - 9D)[y] &= 0\\ D(D^3 - D^2 + 9D - 9)[y] &= 0\\ D(D^2 + 9)(D - 1)[y] &= 0\end{aligned}$$

Thus, the roots of the characteristic equation are $0, 1, \pm 3i$ and the general solution is

$$y = C_1 + C_2 e^x + C_3 \sin 3x + C_4 \cos 3x.$$

. 15 4. Set up **but do not solve** the system of differential equations, including initial conditions, for the following problem. Be sure to define your variables, including units.

Two full tanks, A and B, each holding 50L of liquid, are connected by two pipes. Tank A starts with 25 kg of salt and has pure water flowing in at a rate of 3 L/min. The well mixed solution of tank A flows into tank B at a rate of 4 L/min and in the other direction, the well mixed solution of tank B flows into tank A at rate of 1 L/min. Tank B starts with 10kg of salt and, besides the solution flowing to and from tank A, has 3 L/min (of well mixed solution) draining out.

Solution. Let $A(t)$ be the amount of salt (in kg) in tank A and $B(t)$ be the amount of salt (in kg) in tank B, where t is the time in minutes.

Then we have

$$\begin{aligned}\frac{dA}{dt} &= \text{salt into A} - \text{salt out of A}\\ \frac{dA}{dt} &= 3 \cdot 0 + 1\frac{B}{50} - 4\frac{A}{50}\\ \frac{dA}{dt} &= \frac{B - 4A}{50}\\ \frac{dB}{dt} &= \text{salt into B} - \text{salt out of B}\\ \frac{dB}{dt} &= 4\frac{A}{50} - 4\frac{B}{50}\\ \frac{dB}{dt} &= \frac{2(A - B)}{25}\end{aligned}$$

Finally, the initial conditions are $A(0) = 25$ and $B(0) = 10$.

. 25 5. Using variation of parameters, find the general solution of $y'' + y = \sin^2 t$.

Solution. Writing $y'' + y = 0$ as $(D^2 + 1)[y] = 0$, its general solution is $C_1 \sin t + C_2 \cos t$. We look for a particular solution of the form $u_1 \sin t + u_2 \cos t$.

According to the form, we must solve the equations

$$\begin{aligned}u_1' \sin t + u_2' \cos t &= 0\\ u_1' \cos t - u_2' \sin t &= \sin^2 t\end{aligned}$$

Multiplying the first equation by $\sin t$, the second by $\cos t$ and adding gives

$$u_1'(\sin^2 t + \cos^2 t) = \sin^2 t \cos t.$$

Substituting this into the first equation gives

$$u_2' = \frac{-\sin^3 t \cos t}{\cos t} = -\sin^3 t.$$

Next, we integrate, to obtain

$$\begin{aligned} u_1 &= \int \sin^2 \cos t\, dt \begin{cases} u = \sin t \\ du = \cos t\, dt \end{cases} \\ &= \int u^2 du = \frac{u^3}{3} = \frac{\sin^3 t}{3}. \\ u_2 &= -\int \sin^3\ dt \\ &= -\int (1 - \cos^2 t) \sin t\, dt \begin{cases} u = \cos t \\ du = -\sin t\, dt \end{cases} \\ &= \int (1 - u^2)\, du = u - \frac{u^3}{3} = \cos t - \frac{\cos^3 t}{3}. \end{aligned}$$

Thus, the particular solution is

$$y = \left(\frac{\sin^3 t}{3}\right) \sin t + \left(\cos t - \frac{\cos^3 t}{3}\right) \cos t = \frac{\sin^4 t + \cos^4 t}{3} + \cos^2 t$$

and the general solution is

$$y = C_1 \sin t + C_2 \cos t + \frac{\sin^4 t - \cos^4 t}{3} + \cos^2 t$$

. 20 6. Find the general solution of $y'' - y' = 4t - e^t$.

Solution. The associated homogeneous equation if $y'' - y' = 0$, so $D(D-1)[y] = 0$ and its general solution is $y = C_1 + C_2 e^t$.
The annhilator of $4t$ is D^2 and the annhilator of e^t is $D - 1$. Applying $D^2(D-1)$ to the original DE gives $D^3(D-1)^2[y] = 0$, which has general solution

$$y = C_1 + C_2 t + C_3 t^2 + C_4 e^t + C_5 t e^t$$

Thus, our guess is $y = At + Bt^2 + Cte^t$, which has derivatives

$$\begin{aligned} y' &= A + 2Bt + Cte^t + Ce^t \\ y'' &= 2B + Cte^t + 2Ce^t \end{aligned}$$

Substituting into the DE, we have

$$\begin{aligned} 4t - e^t &= y'' - y' \\ &= (2B + Cte^t + 2Ce^t) - (A + 2Bt + Cte^t + Ce^t) \\ &= (2B - A) - 2Bt - Ce^t \end{aligned}$$

Thus, $2B - A = 0$, $-2B = 4$ and $C = -1$. From the first two equations, we have $B = -2$ and $A = -4$. Thus, the general solution is

$$y = C_1 + C_2e^t - 4t - 2t^2 - te^t.$$