Math 221-601 **Answers to Test #2** **27 July 07**

. 15 1. Consider the IVP $x\frac{dy}{dx} = 1 - y^2$, $y(1) = 0$. Using Euler's method with a step size of 1/3, find (approximately) the value of the solution at $x = 2$.

Solution. Putting the DE in standard form, we have

$$\frac{dy}{dx} = \frac{1 - y^2}{x}.$$

n	x_n	y_n	$f(x_n, y_n)$
0	1	0	1
1	4/3	1/3	2/3
2	5/3	5/9	56/135
3	2	$281/405 \approx .6938$	

Thus, $y(2) \approx .6938$

. 15 2. Solve $xy'' + y' = 4x$.

Solution. This is a reducible DE with y missing, so we substitute $p = \frac{dy}{dx}$ and $p' = \frac{d^2y}{dx^2}$, to get

$$xp' + p = 4x$$

We can either just notice that lefthand side satisfies the product rule, or put it in standard form and find the integrating factor.

$$p' + \frac{p}{x} = 4$$

Now, $\mu = \exp(\int 1/x\,dx) = e^{\ln x} = x$, so

$$\begin{aligned} xp' + p &= 4x \\ \frac{d}{dx}(x \cdot p) &= 4x \\ xp &= \int 4x\,dx = 2x^2 + C \\ p &= 2x + \frac{C}{x} \end{aligned}$$

Then we solve

$$\begin{aligned} \frac{dy}{dx} &= 2x + \frac{C}{x} \\ y &= \int 2x + \frac{C}{x}\,dx = x^2 + C\ln|x| + K \end{aligned}$$

Thus, the general solution is $x^2 + C\ln|x| + K$.

. 25 3. A ball is dropped from a great height and experiences both the acceleration of gravity and a deceleration of $v/2$ due to air resistance.

(a) What is the limiting (terminal) velocity?

(b) When is the velocity 20 ft/s?

Solution. The IVP for this problem (assuming up is the positive direction) is

$$\frac{dv}{dt} = -g - \frac{v}{2}, \qquad v(0) = 0$$

To find the limiting velocity, we solve $-g - v/2 = 0$ to get $v = -2g = -64$ ft/s.

To find when the velocity is 20 ft/s, we must find t so that $v(t) = -20$ and first, we must find v. We have a linear DE for v, so we solve it using an integrating factor.

$$\begin{aligned} \frac{dv}{dt} + \frac{1}{2}v &= -32, & \mu = \exp\left(\int 1/2\,dt\right) = e^{t/2} \\ e^{t/2}\frac{dv}{dt} + \frac{e^{t/2}}{2}v &= -32e^{t/2}\,\frac{d}{dt}\left(e^{t/2}v\right) & = -32e^{t/2} \\ e^{t/2}v &= \int -32e^{t/2}\,dt = -64e^{t/2} + C \\ v &= -64 + Ce^{-t/2} \end{aligned}$$

Using the initial condition $0 = v(0) = -64 + Ce^0 = -64 + C$, we have $C = 64$ and $v(t) = -64 + 64e^{t/2}$ Finally, we solve

$$\begin{aligned} -20 &= -64 + 64e^{-t/2} \\ \frac{44}{64} &= e^{-t/2} \\ \frac{-t}{2} &= \ln(44/64) \end{aligned}$$

and so $t = -2\ln(44/64) \approx 0.7494$. Thus, the rock is going 20 ft/s at approximately .7494 seconds.

· 25 4. Consider the logistic IVP $\dfrac{dP}{dt} = 4P(8-P)$, $P(0) = 4$.

(a) Sketch the phase diagram for the DE. Based on the diagram, what can you say about the solution of the IVP?

(b) Find the exact solution of the IVP.

Solution. Solving $4P(8-P) = 0$ gives roots at $P = 0$ and $P = 8$, so the phase line has equilibrium solutions at $P = 0$ and $P = 8$, with solutions increasing for P between 0 and 8 and decreasing for $P < 0$ or $P > 8$. Thus, $P = 0$ is unstable and $P = 8$ is stable. Since $P(0) = 4$, we expect the solution to increase with a horizontal asymptote at $P = 8$.

To find the exact solution, we separate variables and use partial fractions. Separating gives

$$\frac{dP}{P(8-P)} = 4\,dt$$

Next, we use partial fraction on $1/(P(8-P))$:

$$\frac{1}{P(8-P)} = \frac{A}{P} + \frac{B}{8-P}$$
$$1 = A(8-P) + BP$$

Letting $P = 0$, we have $1 = 8A$ so $A = 1/8$ and letting $P = 8$, we have $1 = 8B$ so $B = 1/8$. Thus,

$$\int \frac{dP}{P(8-P)} = \int \frac{1/8}{P}\,dP + \int \frac{1/8}{8-P}\,dP$$
$$= \frac{1}{8}\ln|P| - \frac{1}{8}\ln|8-P| \qquad = \frac{1}{8}\ln\left|\frac{P}{8-P}\right|$$

Thus, integrating both sides of the separated DE gives

$$\frac{1}{8}\ln\left|\frac{P}{8-P}\right| = 4t + C$$
$$\left|\frac{P}{8-P}\right| = C_1 e^{32t}, \text{where } C_1 = e^C$$

Since, for our initial condition $P/(8-P)$ is positive and will always be positive, we can drop the absolute value bars. Further, the inital condition gives $4/(8-4) = C_1 e^0$, so $C_1 = 1$. Thus, we have

$$\frac{P}{8-P} = e^{32t}, \text{where } C_2 = \pm C_1$$
$$P = 8C_2 e^{32t} - Pe^{32t}$$
$$(1 + e^{32t})P = 8e^{32t}$$
$$P = \frac{8e^{32t}}{1 + e^{32t}}$$

. 20 5. Consider the DE $x^2y'' - 2xy' + 2y = 0$.

(a) Show that $y_1 = x^2$ and $y_2 = x$ are both solutions of this DE.

(b) Show that these two functions are linearly independent.

(c) Find the solution to this DE that satisfies $y(1) = 5$, $y'(1) = 8$.

Solution. To check that $y_1 = x^2$ is a solution, we compute $y_1' = 2x$ and $y_1'' = 2$ and substitute these into the DE to obtain

$$x^2y_1'' - 2xy_1' + 2y_1 = x^2(2) - 2x(2x) + 2(x^2) = 2x^2 - 4x^4 + 2x^2 = 0$$

To check that $y_2 = x$ is a solution, we compute $y_2' = 1$ and $y_1'' = 0$ and substitute these into the DE to obtain

$$x^2y_2'' - 2xy_2' + 2y_2 = x^2(0) - 2x(1) + 2(x) = 0 - 2x + 2x = 0$$

Thus, y_1 and y_2 are solutions.

To see that the functions are linearly independent we use the definition: $x/x^2 = 1/x$ which is clearly not a constant.
The general solution is $y = C_1x^2 + C_2x$, so $y' = 2C_1x + C_2$. Using the initial conditions

$$5 = y(1) = C_1 + C_2, \qquad 8 = 2C_1 + C_2.$$

Taking the second equation minus the first gives $8 - 5 = C_2$ and so $C_2 = 3$. Next, from the first equation $5 = C_1 + 3$ so $C_1 = 2$. Thus, the solution satisfying the given initial conditions is $y = 2x^2 + 3x$.