Math 221-601 **Answers to Test #1** **20 July 07**

. 10 1. Does an Existence and Uniqueness Theorem apply to $y'\sqrt{y} = ty$, $y(0) = 0$? Why or why not? If yes, what does the theorem say about the solution of the IVP.

Solution. The DE is not linear, so we have to use the theorem for general first order DEs.

First, we put the equation in standard form, as

$$y' = t\sqrt{y}$$

The function $f(t, y) = t\sqrt{y}$ is continuous when $t = 0$ and $y = 0$, but $D_y f = t/(2\sqrt{y})$ is not continuous when $y = 0$. Thus, the theorem does not apply.

. 20 2. Solve $2xy' + y = 10\sqrt{x}$, $y(1) = 7$.

Solution. Notice that the DE is linear, so we can put it in standard form as

$$y' + \frac{1}{2x}y = \frac{5}{\sqrt{x}}.$$

The integrating factor is

$$\mu = \exp\left(\int \frac{1}{2x}x\,dx\right) = \exp((\ln x)/2) = \exp(\ln(x^{1/2})) = x^{1/2}.$$

Multiplying through by the integrating factor gives

$$x^{1/2}y' - \frac{1}{2x^{1/2}}y = 5.$$

Notice that the left-hand side of the equation is the product rule applied to $x^{1/2}y$, so we have

$$\frac{d}{dx}\left(x^{1/2}y\right) = 5$$

$$x^{1/2}y = \int 5\,dx = 5x + C$$

Thus, the general solution is $y(x) = 5x^{1/2} + Cx^{-1/2}$. Using the initial condition,

$$7 = y(1) = 5 \cdot 1 + C \cdot 1$$

so $C = 2$ and the solution to the IVP is $y = 5x^{1/2} + 2x^{-1/2}$.

. 20 3. Solve $(xe^y - \sin y)\frac{dy}{dx} = -e^y + 1$.

Solution. This DE is not linear (there's a $\sin y$ in it) or separable, so we put it in differential form and check if it is exact.

Multiplying by dx and rearranging, we have

$$(xe^y - \sin y)dy + (e^y - 1)dx = 0.$$

So, for the check, we have

$$\frac{\partial}{\partial x}(xe^y - \sin y) = e^y, \qquad \frac{\partial}{\partial y}(e^y - 1) = e^y.$$

Since these are equal, the DE is exact.
To find the solution, $F(x, y) = 0$, we first solve

$$\begin{aligned} \frac{\partial F}{\partial y} &= xe^y - \sin y, \\ F &= xe^y + \cos y + g(x). \end{aligned}$$

To find $g(x)$, we next solve

$$\begin{aligned} \frac{\partial F}{\partial x} &= e^y - 1 \\ \frac{\partial}{\partial x}(xe^y + \cos y + g(x)) &= e^y - 1 \\ e^y + g'(x) &= e^y - 1 \end{aligned}$$

Thus, $g'(x) = -1$ and so $g(x) = -x + C$.
The solution to the DE is $xe^y + \cos y - x + C = 0$.

. 25 4. A bottle, initially at 40°F, is put in a 70°F room. After 10 minutes, the bottle is 48°F. Using Newton's law of cooling, what is the temperature of the bottle after 25 minutes?

Solution. Let $T(t)$ be the temperature of the bottle, in °F where t is time, measured in minutes. We know that

$$\frac{dT}{dt} = k(70 - T), \qquad T(0) = 40, \qquad T(10) = 48.$$

First, we solve the DE using separation of variables.

$$\begin{aligned} \frac{dT}{dt} &= k(70 - T) \\ \frac{dT}{70 - T} &= k\,dt \\ \int \frac{dT}{70 - T} &= \int k\,dt \\ -\ln|70 - T| &= kt + C \end{aligned}$$

and, since $70 - T \geq 0$, this simplifies to

$$\begin{aligned} -\ln(70 - T) &= kt + C \\ \ln(70 - T) &= -kt - C \\ 70 - T &= e^{-kt-C} = Ke^{-kt} \text{ where } K = e^{-C} \\ T &= 70 - Ke^{kt} \end{aligned}$$

To find the values of k and K we use the initial conditions:

$$40 = T(0) = 70 - Ke^{k0} = 70 - K$$

so $K = 30$. Next, using $T(10) = 48$, we have

$$\begin{aligned} 48 &= T(10) = 70 - 30e^{k10} \\ -22 &= -30e^{k10} \\ \frac{11}{15} &= e^{k10} \\ -\frac{1}{10}\ln\left(\frac{11}{15}\right) &= k. \end{aligned}$$

Thus, $k \approx -.03101$ and $T(t) = 70 - 30e^{-.03101t}$. The temperature of the bottle after 25 minutes is

$$T(25) = 70 - 30e^{-.03101\cdot 25} = 56.18^\circ F.$$

. 25 5. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min and the (perfectly mixed) solution leaves the tank at 3 gal/min. Notice that the tank will be empty after 60 minutes and that the volume of water in the tank after t minutes is $60 - t$. Find the amount of salt in the tank after t minutes.

Solution. Let $S(t)$ be the amount of salt (in lb) in the tank at time t (in minutes). Working out how much salt is coming in and going out of the tank gives the following IVP:

$$\frac{dS}{dt} = 2 - \frac{3S}{60-t}, \qquad S(0) = 0$$

Rearranging, we have

$$\frac{dS}{dt} + \frac{3}{60-t}S = 2$$

This is a linear DE, and the integrating factor is

$$\mu = \exp\left(\int \frac{3}{60-t}\,dt\right) = \exp\left(-3\ln(60-t)\right) = (60-t)^{-3}.$$

Multiplying by the integrating factor, we have

$$\begin{aligned} (60-t)^{-3}\frac{dS}{dt} + 3(60-t)^{-4}S &= 2(60-t)^{-3} \\ \frac{d}{dt}\left((60-t)^{-3}S\right) &= 2(60-t)^{-3} \end{aligned}$$

Since the product rule checks out, we integrate both sides, giving

$$\begin{aligned} (60-t)^{-3}S &= \int 2(60-t)^{-3}\,dt = (60-t)^{-2} + C \\ S &= (60-t) + C(60-t)^3 \end{aligned}$$

Using the initial condition, we have $0 = 60 + C(60)^3$ and so $C - 1/3600$. Thus, the function is $S(t) = (60-t) + (60-t)^3/3600$.