Math 221-601 — Answers to Review for Test #1 — 18 July 07

. 13 1. For each of the following IVPs, does an appropriate Existence and Uniqueness Theorem apply? Why or why not? If yes, what does the theorem say about the solution of the IVP.

(a) $ty' + y = 2, \quad y(1) = 0$

(b) $y' = (y^2 + t^2)^{1/3}, \quad y(0) = 0$

Solution. For (a), we put the equation in standard form, as

$$y' + \frac{1}{t}y = \frac{2}{t}.$$

The functions $1/t$ and $2/t$ are undefined at $t = 0$ and this is their only discontinuity. Since our initial condition is at $t = 1$, the existence-and-uniqueness theorem for linear IVPs applies and we are guaranteed a solution for all t with $0 < t < +\infty$.

For (b), the DE is clearly not linear, so have to use the general existence and uniqueness theorem. The function $f(t, y) = (y^2 + t^2)^{1/3}$ is continuous at $t = 0$, $y = 0$. However, its partial derivative is

$$\frac{\partial f}{\partial y} = \frac{1}{3}(y^2 + t^2)^{-2/3} 2y,$$

which is not defined for $y = 0$ and $t = 0$. Thus, the theorem does not apply.

. 12 2. Solve $(e^x \sin y - 3x^2)dx + (e^x \cos y + y^{-2/3}/3)dy = 0$.

Solution. This is an exact DE. The solution is $e^x \sin y - x^3 + y^{1/3} = C$.

. 15 3. Consider the IVP $y' = y^2$, $y(0) = 1$.

(a) Verify (showing your work) that $y(t) = 1/(1 - t)$ is a solution of the IVP.

(b) Using Euler's method with step size 1/2, find approximations to the solution at $t = 1/2$ and $t = 1$.

(c) What is the error between the exact and approximate solutions at $t = 1/2$ and $t = 1$?

Solution. To see that $y(t) = 1/(1 - t)$ is a solution, we note that

$$\text{LHS} = \left(\frac{1}{1-t}\right)' = -\frac{-1}{(1-t)^2} = \frac{1}{(1-t)^2} = y^2 = \text{RHS}.$$

Applying Euler's method with $\Delta t = 1/2$, $y_0 = 1$ and $t_0 = 0$, we have

n	t_n	y_n	$y'(t_n, y_n)$	Δy
0	0.0	1	1	0.5
1	0.5	1.5	2.25	1.125
2	1.0	2.625		

Thus, the approximations are $y(1/2) = 1.5$ and $y(1) = 2.625$.
Since the exact values are $y(1/2) = 2$ and $y(1)$ is undefined, the error is .5 at $t = 1/2$ and infinite at $t = 1$. The problem is that error in Euler's method causes it to miss the vertical asymptote in the correct solution.

. 15 4. Solve $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{y^2 + x^2y^2}}, \qquad y(0) = 2.$

Solution. We have

$$\frac{dy}{dx} = \frac{x}{y^3\sqrt{1+x^2}}$$
$$y\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

Integrating both sides, we have

$$\frac{1}{2}y^2 = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

Using the substitution $u = 1 + x^2$, we get

$$\frac{1}{2}y^2 = (1+x^2)^{1/2} + C$$

Solving for y, we have $y = (2(1+x^2)^{1/2} + C')^{1/2}$, where $C' = 2C$. Using the initial condition, we have $2 = (2(1+0)^{1/2} + C')^{1/2} = (2 + C')^{1/2}$. Thus, $4 = 2 + C'$ and so $C' = 2$.
So the solution is $y = (2(1+x^2)^{1/2} + 2)^{1/2}$.

. 20 5. Consider a tank containing 50 L of water with 10 kg of salt dissolved in the water. Salt water containing K kg/L of salt (where K is a constant) are pumped into the tank at a rate of 5 L/min and well-mixed water is drained from the tank at the same rate.

(a) Set up an IVP for this situation.

(b) If there is to be 7 kg of salt in the tank after 10 minutes, what should the value of K be? (Hint: since K is a constant, you can solve the IVP using standard methods.)

(c) For your value of K, what is the eventual amount of salt in the tank? That is, what is the limit as $t \to \infty$?

Solution. Let $S(t)$ be the amount of salt (in kg) in the tank at time t. For part (a), we have

$$\frac{dS}{dt} = 5K - \frac{S}{10}, \qquad S(0) = 10$$

Rearranging, we have

$$\frac{dS}{dt} + \frac{S}{10} = 5K$$
$$e^{t/10}\frac{dS}{dt} + \frac{e^{t/10}}{10}S = 5Ke^{t/10}$$
$$\frac{d}{dt}\left(e^{t/10}S\right) = 5Ke^{t/10}$$

Since the product rule checks out, we integrate both sides, giving

$$\begin{aligned} e^{t/10}S &= 50Ke^{t/10} + C \\ S &= 50K + Ce^{-t/10} \end{aligned}$$

Using the initial condition, we have $10 = 50K + Ce^0$ and so $C = 10 - 50K$. Thus, the function is $S(t) = 50K + (10 - 50K)e^{-t/10}$. Since $S(10) = 7$, we have

$$\begin{aligned} 7 &= 50K + (10 - 50K)e^{-1} \\ 7 &= 50K(1 - e^{-1}) + 10e^{-1} \\ 50K &= \frac{7 - 10e^{-1}}{1 - e^{-1}} \approx 5.254 \end{aligned}$$

Thus, $K = .1051$.
Finally, $\lim_{t\to\infty} S(t) = 50K = 5.254$ kg.

. 25 6. We model a population of mice on an island using an exponential model. After 10 years on the island, the population doubles to 50,000 mice.

(a) Set up an IVP for the population of mice and find a function $M(t)$ for the population of mice since they were introduced to the island.
(b) At this point (i.e., 10 years after the mice arrive), the islanders introduce some cats, who eat 6,000 mice each year. Set up a new IVP for the population of mice after the cats arrive.
(c) Solve the IVP from the previous part and compare the population 10 years after the cats arrive to what it would have been without the cats.
(d) Sketch a direction field for your DE (hint: the right hand side of the DE doesn't depend on t) and predict the behavior of the system.
(e) (bonus question) Is the DE a reasonable population model? Why or why not?

Solution. Let $M(t)$ be the population of the mice, in thousands, measuring from the time in years since the mice arrive. Then the IVP is $\frac{dM}{dt} = kM, \quad M(0) = 25$. Now, we use the general form of the solution, $M(t) = M(0)e^{kt}$ to find k. Since $M(10) = 50$, we have $50 = 25e^{k10}$, and so $k = \ln(2)/10 = .06931$. Thus, the IVP is

$$\frac{dM}{dt} = .06931M, \quad M(0) = 25.$$

The IVP once the cats arrive is

$$\frac{dM}{dt} = .06931M - 6, \quad M(0) = 50.$$

Solving this IVP, we have

$$\begin{aligned} \frac{dM}{dt} &= .06931(M - 86.56) \\ \frac{dM}{M - 86.56} &= .06931\,dt \end{aligned}$$

And integrating gives

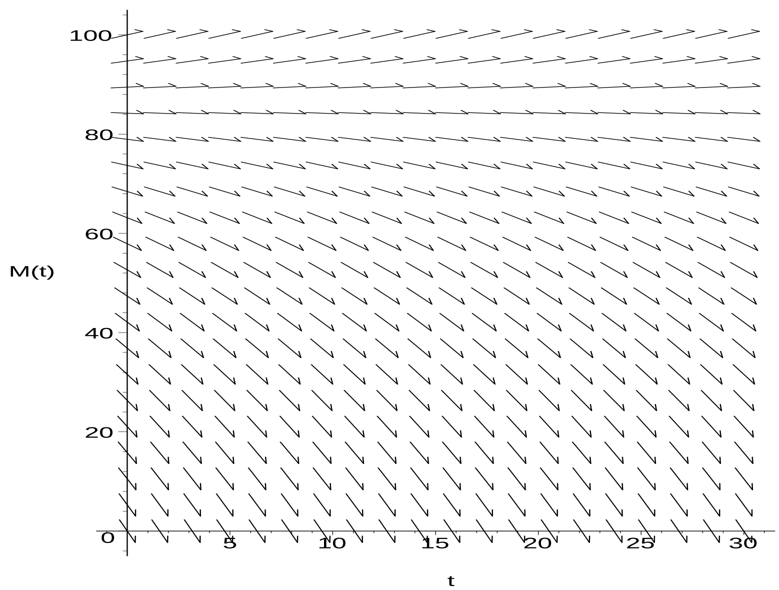
$$\ln(M - 86.56) = .06931t + C$$
$$M - 86.56 = Ke^{.06931t} \qquad \text{where } K = e^C$$
$$M = 86.56 - Ke^{.06931t}$$

Finally, using the inital condition, we have $50 = 86.56 - Ke^{.6931}$, so $K = 36.56/e^{.6931} = 18.28$. Thus,

$$M(t) = 86.56 - 18.28e^{.06931t}$$

and so $M(20) = 13.43$. Since the population of mice without cats doubled in ten years, it will double again, to give a population of mice (without cats) of 100.

The direction field is



Since our starting population is below 86, 560, the population will decrease and eventually become negative!

There are several problems with the model: 1) we don't have an equilibrium solution at zero, 2) the number of mice caught by the cats should depend, in some way, on the number of mice, 3) If the number of mice was large, over 86,560, then the population of mice would grow unboundedly large. So it is not a very good model.