A guide to solving linear systems

The key to solving linear systems is to find eigenvalues and eigenvectors for the matrix. If A is a square matrix, then we say λ is an *eigenvalue* for A if there is a non-zero vector $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$. We call $\vec{v}$ an *eigenvector* for A and λ.

How to find eigenvalues. Rewriting the equation above, $(A - \lambda I)\vec{v} = \vec{0}$, where I is the identity matrix. For any matrix B, $B\vec{v} = \vec{0}$ for a non-zero vector $\vec{v}$ if and only if the determinant of B is zero (this is the key fact about matrices mentioned earlier).

Find eigenvalues for A by solving $det(A - \lambda I) = 0$ for λ.

How to find eigenvectors. Given an eigenvalue for A, say λ_0, then we can plug in λ_0 into the matrix $A - \lambda_0 I$, and this is now a matrix of numbers.

Find eigenvectors for A and λ_0 by solving $(A - \lambda_0 I)\vec{v} = 0$ for non-zero $\vec{v}$.

This works for either complex or real eigenvalues. There will be more than one solution, so pick a simple solution.

Solutions to linear systems. There are three cases: real eigenvalues with 'enough' linearly independent eigenvectors, complex eigenvalues, and real eigenvalues without 'enough' linearly independent eigenvectors.

If we have a real eigenvalue λ_0 which occurs m times in the characteristic equation and we can find m linearly independent eigenvectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then the m linearly independent solutions are

$$\vec{v}_1 e^{\lambda_0 t}, \vec{v}_2 e^{\lambda_0 t}, \ldots \vec{v}_n e^{\lambda_0 t}.$$

If the eigenvalues are $a + bi$ and $a - bi$, then we find the eigenvector for **one** of these eigenvalues, say $a + bi$. The eigenvector will have complex numbers, say

$$\vec{v}_1 = \begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \end{bmatrix}.$$

One way to write a solution is

$$\begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \end{bmatrix} e^{(a+bi)t}$$

but this gives solutions with complex numbers mixed in. Instead, we use Euler's formula $e^{Ki} = \cos K + i \sin K$ to write $e^{(a+bi)t}$ as $e^{at}\cos(bt) + ie^{at}\sin(bt)$. Then, by taking real and imaginary parts of

$$\begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \end{bmatrix} \left(e^{at}\cos(bt) + ie^{at}\sin(bt)\right),$$

the solutions are

$$\vec{Y}_1(t) = e^{at}\begin{bmatrix} \alpha\cos(bt) - \beta\sin(bt) \\ \gamma\cos(bt) - \delta\sin(bt) \end{bmatrix} \quad \vec{Y}_2(t) = e^{at}\begin{bmatrix} \beta\cos(bt) + \alpha\sin(bt) \\ \delta\cos(bt) + \gamma\sin(bt) \end{bmatrix}.$$

If, for a real eigenvalue λ_0, there are not enough linearly independent eigenvectors, then things get complicated.

Suppose there are m copies of λ_0 as eigenvalues but only $p < m$ linearly independent eigenvectors. We must find $m - p$ more linearly independent solutions. Call $m - p$ the *defect* of the eigenvalue λ_0. Let $k = m = p$.

The key idea is that we must find chains of generalized eigenvectors: a chain of eigenvectors is list of non-zero vectors $\vec{v}_k, \vec{v}_{k-1}, \ldots, \vec{v}_1$ so that

$$\begin{aligned}(A - \lambda_0 I)v_1 &= 0\\ (A - \lambda_0 I)v_2 &= v_1\\ &\vdots\\ (A - \lambda_0 I)v_{k-1} &= v_{k-2}\\ (A - \lambda_0 I)v_k &= v_{k-1}\end{aligned}$$

We call k the length of chain. Notice that v_1 is just an ordinary eigenvalue.

The k linearly independent solutions associated to this chain have the form

$$\begin{aligned}&\vec{v}_1 e^{\lambda_0 t}\\ &\vec{v}_1 t e^{\lambda_0 t} + \vec{v}_2 e^{\lambda_0 t}\\ &\vec{v}_1 \frac{t^2}{2} e^{\lambda_0 t} + \vec{v}_2 t e^{\lambda_0 t} + \vec{v}_3 e^{\lambda_0 t}\\ &\vdots\\ &\vec{v}_1 \frac{t^k}{k!} e^{\lambda_0 t} + \frac{t^{k-1}}{(k-1)!} e^{\lambda_0 t} + \cdots + \vec{v}_{k-1} t e^{\lambda_0 t} + \vec{v}_k e^{\lambda_0 t}\end{aligned}$$

If there is more than one chain for a particular eigenvalue, then there are also linear independence conditions we have to check.

So the method is to solve $(A - \lambda_0 I)^{k+1} \vec{v}_k = 0$. Then we multiply $\vec{v}_k$ by $A - \lambda_0 I$ to get $\vec{v}_{k-1}$, and then multiply $\vec{v}_{k-1}$ by $A - \lambda_0 I$ to get $\vec{v}_{k-2}$ and so on until we have them all. If, at some point, we get one of the $\vec{v}_i$ to be zero, then we have to go back and choose a different $\vec{v}_k$.

For 2×2 systems, there is a particularly nice form: If the eigenvalue is λ_1, with only one linearly independent eigenvector $\vec{v}_1$, then solve $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1$ to get $\vec{v}_2$. So $\vec{v}_2, \vec{v}_1$ is a chain of generalized eigenvalues of length 2. The solutions to the system are

$$\vec{Y}_1(t) = \vec{v}_1 e^{\lambda_1 t}, \qquad \vec{Y}_2(t) = (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda_1 t},$$

At this point, examples are more helpful than further theory.

Examples. Next, we carry out this process for four examples, to show how it works in each case. Consider the system

$$\frac{dx}{dt} = 2x + 2y, \qquad \frac{dy}{dt} = 3x + y.$$

To write this as a matrix, let $\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$. Then we have $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$.

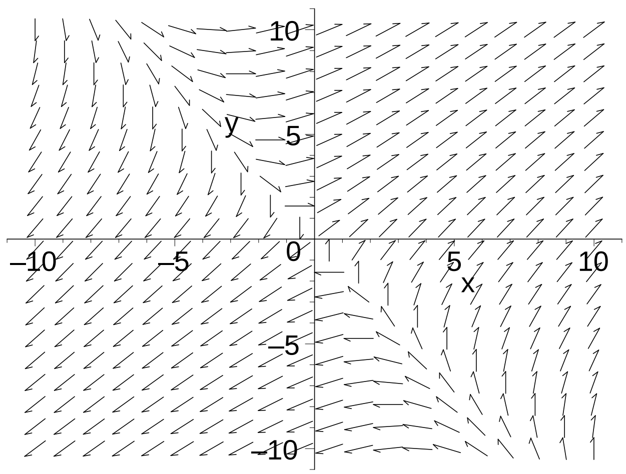
Eigenvalues. We solve $\begin{vmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 3\lambda - 4 = 0$. The roots are $\lambda = -1$ and $\lambda = 4$.

Eigenvectors. First we find the eigenvector for $\lambda = 4$. Solve $\begin{bmatrix} 2-4 & 2 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $-2a+2b = 0$ and $3a-3b = 0$. **It will always be true that the two equations are multiples of each other. If this does not happen, then you've made a mistake somewhere.** Pick a simple solution, like $a = 1$ and $b = 1$. So the eigenvector for $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next, we find the eigenvector for $\lambda = -1$. Solve $\begin{bmatrix} 2-(-1) & 2 \\ 3 & 1-(-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Here, the two equations are $3a + 2b = 0$ and $3a + 2b = 0$, so the equations are not just multiples, but are identical. A simple solution is $a = -2$ and $b = 3$. So the eigenvector for $\lambda = -1$ is $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Thus, the general solution is $\vec{Y}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-t}$. In terms of the component functions, $x(t)$ and $y(t)$, we have $x(t) = C_1 e^{4t} - 2C_2 e^{-t}$ and $y(t) = -C_1 e^{4t} + 3C_2 e^{-t}$.

Phase plane. The phase plane of this system is



Notice the line that is multiples of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is, the line $y = x$. On this line, solutions move straight out, away from the origin. This is because the eigenvector gives us a straight line of solutions through the origin on the phase plane. Moreover, because the associated eigenvalue is positive, the solutions move away from the origin.

The other eigenvector, $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, also gives us a straight line of solutions through the origin, now on the line $y = -3x/2$. Because the eigenvalue is negative, the solutions more towards the origin on this line.

If the two eigenvalues are positive and distinct, then solutions move away from the origin along both straight line solutions. If the two eigenvalues are negative and distinct, then solutions move towards the origin along both straight line solutions. (See the pictures at the end of the handout.)

Next, we consider a system which will turn out to have complex eigenvalues,

$$\frac{dx}{dt} = x + 5y, \qquad \frac{dy}{dt} = -x + 3y.$$

It has the form $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with $A = \begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$ and $\vec{Y}(t)$ in the last example.

Eigenvalues. We solve $\begin{vmatrix} 1-\lambda & 5 \\ -1 & 3-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 4\lambda + 8 = 0$. The roots are $\lambda = 2 \pm 2i$.

Eigenvectors. To find this for $\lambda = 2-2i$, solve $\begin{bmatrix} 1-(2-2i) & 5 \\ -1 & 3-(2-2i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $(-1+2i)a + 5b = 0$ and $-a + (1+2i)b = 0$. These equations are still multiples of each other (multiply the first by $(1+2i)/5$ to get the second) but it is maybe more trouble than it's worth to check this when the equations with complex numbers.

To pick a solution we set a equal to the coefficient of b in the equation and b equal to minus the coefficient of a. Thus, $a = 5$ and $b = 1-2i$ is a solution. So the eigenvector for $\lambda = 2-2i$ is $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$.

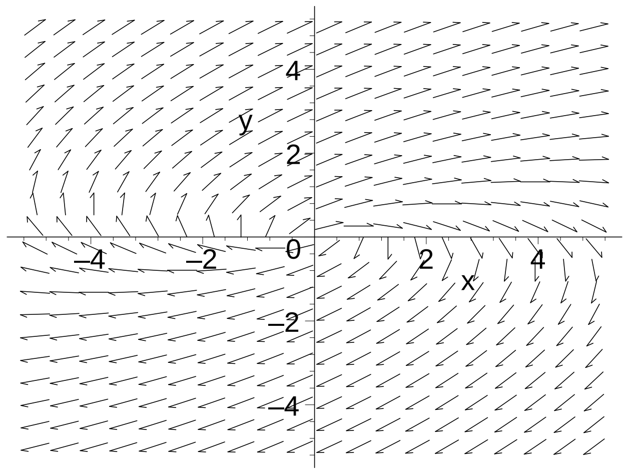
From this one eigenvector, we can find two solutions, using the formula given on the first page. The solutions are

$$\vec{Y}_1(t) = e^{-2t} \begin{bmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} \quad \vec{Y}_2(t) = e^{-2t} \begin{bmatrix} 5\sin(2t) \\ -2\cos(2t) + \sin(2t) \end{bmatrix}.$$

Notice that **we do not need to find the eigenvector for the second complex eigenvalue**. The general solution of the system is

$$\vec{Y}(t) = C_1 e^{-2t} \begin{bmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 5\sin(2t) \\ -2\cos(2t) + \sin(2t) \end{bmatrix}.$$

Phase plane. The phase plane of this system is



Because the eigenvalues are complex instead of real, we get a spiral instead of straight lines of solutions. The real part of the eigenvalue tells us whether the solutions spiral inwards toward the origin or spiral outwards away from the origin. If the real part is negative, they spiral inward; if positive, they spiral outward; if zero, then the solutions loop, staying roughly the same distance from the origin.

For the next example, we consider a solution with a repeated eigenvalue,

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = -x + 3y.$$

It has the form $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$ with $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ and $\vec{Y}(t)$ in the last example.

Eigenvalues. We solve $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 4\lambda + 4 = 0$. The roots $\lambda = 2$, repeated.

Eigenvectors. First we find the eigenvector for $\lambda = 2$. Solve $\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $-a + b = 0$ and $-a + b = 0$, so we happen to have exactly the same equation. Pick a solution $a = 1$ and $b = 1$. So the eigenvector for $\lambda = 2$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the first solution in the fundamental set is

$$\vec{Y}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

There is no second eigenvalue, so we have to find a second linearly independent solution by finding another vector. We solve the system the system given on the first page, namely

$$\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

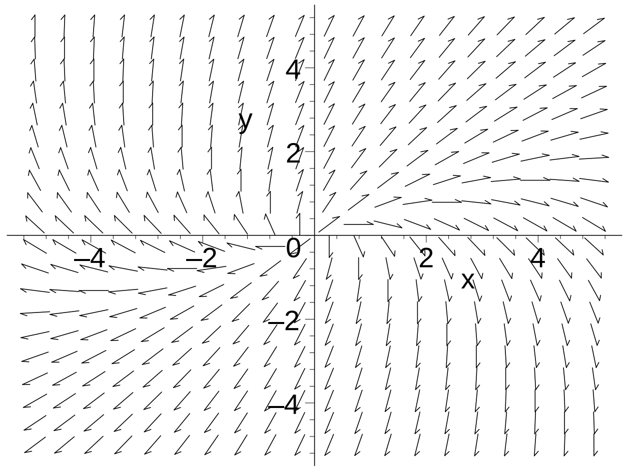
This system is the equations $-a+b=1$ and $-a+b=1$. A simple solution is $a=0$ and $b=1$. According to the formula on the first page, the second solution is

$$\vec{Y}_2(t) = \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}0\\1\end{bmatrix}\right) e^{2t}.$$

The general solution is

$$\vec{Y}(t) = C_1 \begin{bmatrix}1\\1\end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}0\\1\end{bmatrix}\right) e^{2t}.$$

Phase plane. The phase plane of this system is



Because we have only one eigenvalue and one eigenvector, we get a single straight-line solution; for this system, on the line $y=x$, which are multiples of the vector $\begin{bmatrix}1\\1\end{bmatrix}$. Notice that the system has a bit of spiral to it.

For the final example, we consider the system

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix}-3 & 0 & -4\\ -1 & -1 & -1\\ 1 & 0 & 1\end{bmatrix} \vec{Y}$$

By taking the determinant as usual, the eigenvalues are $-1,-1,-1$. When we try to solve $(A-(-1)I)\vec{x}=0$, we get

$$\begin{bmatrix}-2 & 0 & -4\\ -1 & 0 & -1\\ 1 & 0 & 2\end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Taking the -1 times the second equation and the third equation, we get

$$x_1 + x_3 = 0$$
$$x_1 + 2x_3 = 0$$

Solving this gives $v_1 = v_3 = 0$, so there is only one eigenvector,

$$\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Thus, the eigenvalue -1 has defect 2 and start by trying to solve $(A+I)^3\vec{v}_3 = 0$. It turns out that

$$(A+I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so we can pick

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or any other non-zero vector, for that matter.

Then

$$\vec{v}_2 = (A+I)\vec{v}_3 = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

and

$$\vec{v}_1 = (A+I)\vec{v}_2 = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\vec{v}_2$ and $\vec{v}_1$ are both non-zero, our choice for $\vec{v}_3$ worked.

Finally, we can put this together to get three linearly independent solutions:

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) e^{-t}$$

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right) e^{-t}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

Exercises. Repeat the analysis done in the examples for the following matrices.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$$

To get maple to plot the phase plane of a DE, you can do the following

```
with(DEtools);
sys := {diff(x(t),t) =1*x(t)+1*y(t), diff(y(t),t)=-x(t)+3*y(t)};
DEplot(sys, [x(t),y(t)], t=-10..10, y=-5..5,x=-5..5);
```