1. The base of a solid is the region between the curve \( y = 2\sqrt{\sin x} \) and the interval \([0, \pi]\) on the \( x \)-axis. If the cross-sections perpendicular to the \( x \)-axis are semi-circles with diameters running from the \( x \)-axis to the curve \( y = 2\sqrt{\sin x} \), then find the volume of this solid.

Draw a suitable diagram.

Solution. This is a variation on problem 5 in Section 6.1 (semi-circular cross-sections instead of triangles or squares), an assigned homework problem.

For a slice at a fixed value of \( x \), the endpoints of the base in the \( x,y \)-plane are \((x, 0)\) and \((x, 2\sqrt{\sin x})\). The area of a semi-circle \( \pi r^2/2 \), so we need to find the radius, which is half the distance from \((x, 0)\) to \((x, 2\sqrt{\sin x})\), i.e., \( \sqrt{\sin x} \). Thus, the area of each slice is \( (\pi \sin x)/2 \).

The limits of integration are \( x = 0 \) and \( x = \pi \).

The volume of the solid is

\[
V = \int_{0}^{\pi} \frac{\pi \sin x}{2} \, dx = \frac{\pi}{2} \int_{0}^{\pi} \sin x = \frac{\pi}{2} \left. -\cos x \right|_{0}^{\pi} = \frac{\pi}{2} (-\cos \pi + \cos 0) = \pi.
\]

2. Using suitable substitutions, evaluate the following integrals:

(a) \( \int \frac{1}{x \ln x} \, dx \)  
(b) \( \int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^{x} \cos e^{x} \, dx \)

Solution. Part (a) is Problem 43 from Section 5.5, an assigned homework problem; part (b) is Problem 23 on page 380.

For (a), we use the substitution \( u = \ln x \), with \( du = \frac{1}{x} \, dx \), to get the indefinite integral

\[
\int \frac{1}{u} \, du = \ln |u| + C = \ln |\ln x| + C
\]

For (b), we use the substitution \( u = e^{x} \), with \( du = e^{x} \, dx \). Notice that if \( x = \ln(\pi/6) \), then \( u = \pi/6 \) and if \( x = \ln(\pi/2) \) then \( u = \pi/2 \). Thus

\[
\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^{x} \cos e^{x} \, dx = \int_{\pi/6}^{\pi/2} 2 \cos x \, dx = 2 \sin x|_{\pi/6}^{\pi/2} = 2 \sin(\pi/2) - 2 \sin(\pi/6) = 1.
\]

3. Set up the integral(s) for the following area BUT DO NOT evaluate the integral(s).

The area of the region in the first quadrant bounded on the left by the \( y \)-axis, below by the curve \( x = 2\sqrt{y} \), above left by the curve \( x = (y - 1)^2 \) and above right by the line \( x = 3 - y \).
Solution. This is the problem from Homework 10, except you don’t have to evaluate the integral.

Notice that the intersection points of \( x = 2\sqrt{y} \) and \( x = 3 - y \) is \((2,1)\) as can be checked by substituting the point into both curves. Similarly the intersection point of \( x = (y - 1)^2 \) and \( x = 3 - y \) is \((1,2)\).

For \( y \) from 0 to 1, the upper endpoint is \((2\sqrt{y}, y)\) and the lower endpoint is \((0,y)\), so the length is \(2\sqrt{y}\). For \( y \) from 1 to 2, the upper endpoint is \((3-y, y)\) and the lower endpoint is \(((y - 1)^2, y)\), so the length is \(3 - y - (y - 1)^2 = 2 + y - y^2\). Thus, the area is \(\int_0^1 2\sqrt{y} \, dy + \int_1^2 \left(2 + y - y^2\right) \, dy\).

4. Use l’Hôpital’s Rule to evaluate \( \lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta} \).

Be sure to justify the use of l’Hôpital’s Rule.

Solution. This is Problem 19 from Section 4.6, an assigned homework problem.

Notice that \(1 - \sin(\pi/2) = 0\) and \(1 + \cos(\pi) = 0\), so this limit is a 0/0 indeterminate form and we can apply l’Hôpital’s Rule.
\[
\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta} = \lim_{\theta \to \pi/2} \frac{-\cos(\theta)}{-2 \sin \theta} = 0
\]
\[
\lim_{\theta \to \pi/2} \frac{-\cos(\pi/2)}{-2 \sin(\pi/2)} = 0
\]
\[
= \lim_{\theta \to \pi/2} \frac{\sin(\theta)}{-4 \cos 2\theta}
\]
\[
= \frac{\sin(\pi/2)}{-4 \cos(\pi/2)} = \frac{1}{4}.
\]

5. Consider the function \( f(x) = 3 - x^2 \) on the interval \([0, 1]\). Set up in \( \Sigma \)-notation but do not evaluate the Riemann sum for this function using a partition of \([0, 1]\) into \( n \) equal subintervals and the right-hand rule.

**Solution.** This is part of Homework 9 with the function \( 3x^2 \) replaced by \( 3 - x^2 \) and the interval \([0, 2]\) by \([0, 1]\).

We divide the interval \([0, 1]\) into \( n \) intervals
\[
\left[ 0, \frac{1}{n} \right], \left[ \frac{1}{n}, \frac{2}{n} \right], \ldots, \left[ \frac{n-1}{n}, \frac{n}{n} \right].
\]

So, for each “rectangle” we have a base of \( 1/n \).

We are evaluating the function at right-hand endpoint of each interval, i.e., \( 1/n \) on \([0, 1/n]\), \( 2/n \) on \([1/n, 2/n]\), and so on. Thus, the formula for the \( k \)th term, which is for the interval \([\frac{k-1}{n}, \frac{k}{n}]\), is
\[
f \left( \frac{k}{n} \right) \cdot \frac{1}{n} = \left( 3 - \frac{k^2}{n^2} \right) \frac{1}{n} = \frac{3}{n} - \frac{k^2}{n^3}\]

The Riemann sum then is \( \sum_{k=1}^{n} \frac{3}{n} - \frac{k^2}{n^3} \).

6. You are designing a rectangular poster to contain 100 in\(^2\) of printing with an 8 inch margin at top and bottom and 2 in margin at each side. What overall dimensions for the piece of paper will minimize the amount of paper used?

**Solution.** This is a variation on Problem 11 from Section 4.5, an assigned homework problem.

Let the height and width of the poster be \( x \) and \( y \) respectively. We want to minimize the area, which is \( A = xy \).

Then the area available for printing is \( (x - 16)(y - 4) \), and this has to be 100 sq. in., so \( (x - 16)(y - 4) = 100 \). Solving this equation for \( y \), we have \( y = 4 + 100/(x - 16) \). Substituting this into the the formula, we have
\[
A(x) = x(4 + \frac{100}{x - 16}) = 4x + \frac{100x}{x - 16}.
\]
Notice that $x$ must be greater than 16 and can be as large as we like, so the domain is $(16, +\infty)$. Differentiating, we have

$$A' = 4 + \frac{(x-16)100 - 100x}{(x-16)^2} = 4 - \frac{1600}{(x-16)^2} = \frac{4(x-16)^2 - 1600}{(x-16)^2}.$$ 

In order for the derivative to be zero, we must have

$$4(x - 16)^2 - 1600 = 0$$

$$(x - 16)^2 = 400$$

$$(x - 16) = 20$$

$$x = 36$$

and

$$y = 4 + \frac{100}{36 - 16} = 9$$

To see that this answer is indeed a minimum, observe that

$$A''(x) = \frac{d}{dx} \left( 4 - \frac{1600}{(x - 16)^2} \right) = \frac{3200}{(x - 16)^3}$$

and so $A''(36) > 0$, so by the second derivative test, $A(x)$ has a minimum at $x = 36$. 