1. Consider the straight line given by \( f(x) = mx + b \) where \( m \neq 0 \).

(a) Give a convincing (algebraic) argument of why \( f \) is a one-to-one function. Your argument does not need to be long, but it does need to use the definition of one-to-one.

(b) Find a formula for the inverse of \( f \).

(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines, each with a nonzero slope, then what can you say about the graphs of the inverses of the functions?

Solution. For (a), the definition says that, for a function \( f(x) \) to one-to-one, if \( f(x_1) = f(x_2) \), then we must have \( x_1 = x_2 \). So suppose \( f(x_1) = f(x_2) \), that is, by the definition of \( f(x) \),

\[
mx_1 + b = mx_2 + b
\]

Subtracting \( b \) from both sides gives

\[
x_1 = x_2.
\]

Since \( m \neq 0 \), we can divide both sides by \( m \) to get

\[
x = x_1.
\]

This shows that if \( f(x_1) = f(x_2) \), we must have \( x_1 = x_2 \), that is, the only way two \( x \)-values can produce the same value of \( f(x) \) is if the two \( x \)-values are equal.

For (b), we switch \( x \) and \( y \) and solve for \( y \). First,

\[
x = my + b
\]

Subtracting \( b \) from both sides gives

\[
x - b = my
\]

Since \( m \neq 0 \), we can divide both sides by \( m \) to get

\[
x - b = my
\]

Thus, \( y = x/m - b/m \). So the formula for \( f^{-1}(x) \) is

\[
f^{-1}(x) = \frac{x}{m} - \frac{b}{m}
\]

That is, the inverse function is a line with slope \( 1/m \) and \( y \)-intercept \(-b/m\).

For (c), notice that parallel lines have the same slope. Call this slope \( m \). Then the inverse functions of these two lines will both have slope \( 1/m \) from part (b). So the graphs of the inverses of the functions will also be parallel lines.

For (d), if the two lines are perpendicular, then one will have slope \( m \) and the other will have slope \(-1/m\). The corresponding inverse functions will be lines with slopes \( 1/m \) and \( 1/(-1/m) = -m \). If we take the negative of the reciprocal of \( 1/m \), we get \(-m \), so the two inverse functions are perpendicular lines, just like the ones we started with.