1. Find \( f'(x) \) for the following functions. (You need not simplify your answers.)

(a) \( f(x) = \frac{(4x + 1)^4}{3x^2 - 1} \)

(b) \( f(x) = (x^2 + 4)^4 e^{5x^2 + 1} \)

(c) \( f(x) = e^{-x^3 + x} + \ln(3e^{x^2} - 2x) \)

\[ \text{Solution.} \]

For part (a), we use the quotient rule to get

\[
f'(x) = \frac{(3x^2 - 1)4(4x + 1)^3 \cdot 4 - (4x + 1)^4 \cdot 4 - (4x + 1)^4 \cdot 6x}{(3x^2 - 1)^2}.
\]

For part (b), we use the product rule and then the chain rule to get

\[
f(x) = 4(x^2 + 4)^3 \cdot 2x \cdot e^{5x^2 + 4} + (x^2 + 1)^4 \cdot e^{5x^2 + 1} \cdot 10x.
\]

For part (c), we use the rules for \( e^{h(x)} \) and \( \ln h(x) \) to get

\[
f'(x) = e^{-x^3 + x}(-3x^2 + 1) + \frac{3e^{x^2}(2x) - 2}{3e^{x^2} - 2x}.
\]

2. Find the equation of the tangent line to the graph of the curve \( y = f(x) = (3x^2 - 10)^3 \) at \( x = 2 \).

\[ \text{Solution.} \]

First, we find \( f(2) = (3 \cdot 2^2 - 10)^3 = 2^3 = 8 \) and then \( f'(x) = 3(3x^2 - 9)^2(6x) \) so \( f'(1) = 3(3 \cdot 2^2 - 10)^2(6) = 72 \). Thus, the equation of the tangent line is

\[ y - 8 = 72(x - 2). \]

3. Cesium-137 has a half-life of 30.07 years. How much of a 40 gram mass is left after 50 years?

\[ \text{Solution.} \]

Let \( C(t) \) be the amount of Cesium, in grams, after \( t \) years. Then we know that \( C(0) = 40 \), \( C(30.07) = 20 \) and we have to find \( C(50) \).
The general formula for \( C(t) \) is \( Ae^{kt} \) and to find \( A \) we notice that \( 40 = C(0) = Ae^{k \cdot 0} = A \). To find \( k \), we solve

\[
20 = C(2.065) = 40e^{k \cdot 30.07}
\]

\[
\frac{1}{2} = e^{k \cdot 30.07}
\]

\[
\ln \left( \frac{1}{2} \right) = k \cdot 30.07
\]

\[
k = \frac{\ln \left( \frac{1}{2} \right)}{30.07} = -0.0231
\]

Thus, \( C(50) = 40e^{-0.0231 \cdot 50} = 12.63 \) grams.

4. Solve the following equations for \( x \):

(a) \( 3e^{2x + 10} = 18 \)

(b) \( 27^{x-1} = 3^{4x} \)

**Solution.** For part (b), we have

\[
e^{2x+10} = \frac{18}{3} = 6
\]

\[
\ln(e^{2x+10}) = \ln 6
\]

\[
2x + 10 = \ln 6
\]

\[
2x = (\ln 6) - 10
\]

\[
x = \frac{(\ln 6) - 10}{2}
\]

For part (a), we have

\[
(2^4)^{-x+1} = (2^3)^{2x}
\]

\[
2^{4(-x+1)} = 2^{6x}
\]

\[
4(-x + 1) = 6x
\]

\[
-4x + 4 = 6x
\]

\[
4 = 10x,
\]

and so \( x = 4/10 \).

5. I need to have $50,000 in 21 years to pay my daughter’s college tuition. How much must I invest now in an account paying 8% compounded annually to have the required amount in 21 years?
Solution. Using the equation

\[ P = A \left(1 + \frac{r}{m}\right)^{mt} \]

with \( P = 50000 \), \( r = .08 \), \( m = 1 \), and \( t = 21 \), we have 50,000 = \( A(1.09)^{21} \) and so \( A = 50,000/(1.09)^{21} \), which gives $9,932.79 as the required amount to invest now.

6. Let \( f(x) = \frac{1}{4}x^4 - 3x^3 + 9x^2 + 2 \), for all numbers \( x \).

(a) Find all critical numbers of \( f(x) \).
(b) Chart \( f'(x) \) on a number line.
(c) List the open intervals on which \( f(x) \) is an increasing function.
(d) List the open intervals on which \( f(x) \) is a decreasing function.

Solution. Notice that

\[ f'(x) = x^3 - 9x^2 + 18x \]
\[ = x(x^2 - 9x + 18) \]
\[ = x(x - 3)(x - 6) \]

Since \( f'(x) \) is never undefined, the only critical numbers are where \( f'(x) = 0 \), that is, 0, 3, and 6.

Since \( f(x) \) is never undefined, the only points where we have to split the number line are 0, 3, and 6. For the interval \((−\infty, 0)\), we pick \( x = -1 \) and get \( f'(-1) = (-1)(-4)(-7) < 0 \). For the interval \((0, 3)\), we pick \( x = 1 \) and get \( f'(1) = 1 \cdot (-2)(-5) > 0 \). For the interval \((3, 6)\), we pick \( x = 4 \) and get \( f'(4) = 4 \cdot 1 \cdot (-2) < 0 \). For the interval \((6, +\infty)\), we pick \( x = 7 \) and get \( f'(7) = 7 \cdot 4 \cdot 1 > 0 \).

Thus, \( f(x) \) is increasing on \((0, 3)\) and \((6, +\infty)\) and it is decreasing on \((−\infty, 0)\) and on \((3, 6)\).

7. If \( \log_b 2 = a \) and \( \log_b 7 = c \), express \( \log_b 98 \) in terms of \( a \) and \( c \).

Solution. Since \( 98 = 2 \cdot 7^2 \), by the properties of logarithms,

\[ \log_b 98 = \log_b(2 \cdot 7^2) \]
\[ = \log_b 2 + \log_b 7^2 \]
\[ = \log_b 2 + 2 \log_b 7 = a + 2c. \]
8. How long does it take (in years) for $30,000 to double in value, if it is invested in an account paying 8% compounded quarterly?

Solution. Using the equation

\[ P = A \left(1 + \frac{r}{m}\right)^{mt} \]

with \( A = 30000 \), \( P = 60000 \), \( r = .08 \), and \( m = 4 \), we have

\[
60000 = 30000 (1.02)^{4t}
\]

\[
2 = (1.02)^{4t}
\]

\[
\ln 2 = \ln (1.02)^{4t}
\]

\[
\ln 2 = 4t \ln 1.02
\]

\[
t = \frac{\ln 2}{4 \ln 1.02} = 8.7506
\]

Thus, it will take 8.7506 years or, to be precise, 9 years. (After 8 years and 3 quarters, the amount is $59,996.69.)