Math 221-601 Answers to Quiz # 2 25 July '07

. 4 1. Sketch the phase diagram (including critical points and whether each critical point is stable or unstable) for the following DE. Then, using the phase diagram, sketch typical solution curves, including the equilibrium solutions.

$$\frac{dx}{dt} = x^2 - 4x + 3$$

Solution. The critical points are the roots of $x^2 - 4x + 3 = (x-3)(x-1)$, that is, $x = 1$ and $x = 3$. Drawing the phase line, we see that x' is decreasing for x between 1 and 3 and increasing for $x < 1$ or $x > 3$. Hence the critical point at $x = 1$ is stable and the one at $x = 3$ is unstable.

There should be a graph of the sample solutions, but I'm not including it here.

. 6 2. A motorboat is moving at 50 ft/s when its motor suddenly stops. The boat slows to 25 ft/s ten seconds later. How far will the boat coast, in all?

Solution. We assume that the velocity of the boat satisfies

$$\frac{dv}{dt} = -kv$$

for some constant k. Next, we know that $v(0) = 50$ and $v(10) = 25$.

Separating variables, we can solve the DE

$$\begin{aligned} \frac{dv}{dt} &= -kv \\ \int \frac{dv}{v} &= -\int k\,dt \\ \ln|v| &= -kt + C \\ v &= Ke^{-kt} \text{ where } K = e^C,\ v \geq 0 \end{aligned}$$

Since $50 = v(0) = Ke^0$, $K = 50$. To find k, notice

$$\begin{aligned} 25 &= 50e^{-k\cdot 10} \\ k &= -\frac{1}{10}\ln(1/2) \approx .0693 \end{aligned}$$

Thus, the position function is

$$\begin{aligned} x(t) = \int_0^t v(s)\,ds &= \int_0^t 50e^{-ks}\,ds \\ &= \left.\frac{50}{-k}e^{-ks}\right|_0^t \\ &= \frac{50}{k}(1 - e^{-kt}) \end{aligned}$$

Notice that the velocity approaches but never equals zero, so the total distance traveled is the limit as t approaches $+\infty$ of $x(t)$, that is

$$\lim_{t\to\infty} \frac{50}{k}(1 - e^{-kt}) = \frac{50}{k}.$$

Since $k = .0693$, the distance is $50/.0693 = 721.35$ ft.