Math 221-601- Differential Equations **2nd 5-week Summer Session, 2007**

Trig Substitution Integrals

Evaluate $\displaystyle\int \frac{3x+11}{x^2+6x+15}\,dx$.

Notice that

$$x^2+6x+15 = x^2+6x+9+6 = (x+3)^2+6$$

so the quadratic denominator is irreducible. Thus, we cannot use partial fractions on this integral.

First, we split the integral into two fractions, so that one is an easy substitution and the other had a constant numerator. Since we're going to be using the substitution $u=(x+3)^2+6$ in the first integral, we split the numerator as

$$3x+11 = 3(x+3)+2$$

and then the integral becomes

$$\int \frac{3x+11}{x^2+6x+15}\,dx = \int \frac{3(x+3)}{(x+3)^2+6}\,dx + \int \frac{2}{x^2+6x+15}\,dx.$$

For the first integral, the subsitution $u=(x+3)^2+6$ gives

$$\int \frac{3(x+3)}{(x+3)^2+6}\,dx = \int \frac{3/2}{u}\,du = \frac{3}{2}\ln|u|+C = \frac{3}{2}\ln\left|(x+3)^2+6\right|+C.$$

For the second integral, we rewrite the fraction as

$$\frac{2}{x^2+6x+15} = \frac{2/6}{\left(\frac{x+3}{\sqrt{6}}\right)^2+1}$$

Using this and the substitution $u=(x+3)/\sqrt{6}$, we obtain

$$\begin{aligned}\int \frac{2}{x^2+6x+15}\,dx &= \int \frac{1/3}{\left(\frac{x+3}{\sqrt{6}}\right)^2+1}\,dx \\ &= \int \frac{\sqrt{6}/3}{u^2+1}\,du \\ &= \frac{\sqrt{6}}{3}\arctan u + C \\ &= \frac{\sqrt{6}}{3}\arctan\left(\frac{x+3}{\sqrt{6}}\right)+C\end{aligned}$$

Putting these together, we obtain

$$\int \frac{3x+11}{x^2+6x+15}\,dx = \frac{3}{2}\ln\left|(x+3)^2+6\right| + \frac{\sqrt{6}}{3}\arctan\left(\frac{x+3}{\sqrt{6}}\right)+C$$