Math 871–872 Qualifying Exam
June 2011

Solve three problems from Section A and three more from Section B; you may work on any number of the problems, but indicate which six you want graded. When in doubt about the wording of a problem, ask for clarification. Do not interpret a problem in such a way that it becomes trivial. **Justify your answers.**

**Section A:**

1. Prove that if \( f : X \to Y \) is a continuous map between topological spaces and \( C \) is a compact subset of \( X \), then \( f(C) \) is a compact subset of \( Y \).

2. Suppose \( f : X \to Y \) is a quotient map. Prove that if \( Y \) is connected and \( f^{-1}(\{y\}) \) is a connected subspace of \( X \) for all \( y \in Y \), then \( X \) is connected.

3. Prove that every metrizable space is normal Hausdorff (aka \( T_4 \)).

4. Suppose \( A, B \) are disjoint, compact subspaces of the Hausdorff topological space \( X \). Prove there are open subsets \( U, V \) of \( X \) such that \( A \subseteq U, B \subseteq V \) and \( U \cap V = \emptyset \).

**Section B:**

5. Let \( X \) be the space obtained by deleting three distinct points from \( \mathbb{R}^2 \). Compute \( \pi_1(X) \).

6. View \( S^3 \) as the set of unit vectors in \( \mathbb{R}^4 \), and consider the equivalence relation on them induced by \( u \sim v \) if \( A \cdot u = v \), where \( A \) is the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 
\end{pmatrix}
\]

Compute the fundamental group of the quotient space \( S^3/\sim \).

7. (i) Describe a way of identifying pairs of faces of \( \Delta^3 \) (the standard three simplex) to produce a \( \Delta \) complex structure on \( S^3 \) having a single 3 simplex.

(ii) Write down the chain complex corresponding to the \( \Delta \)-complex in (i). Be sure to include the differentials of the complex, but you do **not** need to compute the homology.

8. Let \( X \) be the space obtained from the sphere \( S^2 \) by joining the North and South poles together with a straight line segment.

(i) Describe a structure of a CW complex on \( X \).

(ii) Compute the homology of \( X \) using cellular homology, with decomposition from (i).

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