There are 10 questions, of which you should attempt 8. Each question carries equal weight. One complete answer is worth more than two partially completed questions. Standard results may be quoted provided they are clearly stated. All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A graph is \( r \)-regular if every vertex has degree \( r \). \( P_n \) is the path with \( n \) vertices. An induced subgraph of the graph \( G = (V, E) \) is a subgraph of the form \( H = (W, F) \) with \( W \subset V \) and \( F = \{ xy \in E : x, y \in W \} \). Given a graph \( G \) its clique number, \( \omega(G) \), is the number of vertices in the largest complete subgraph of \( G \). A bridge in a connected graph is an edge whose removal disconnects the graph.

**Question 1.** What is the general solution to the recurrence
\[
x_n + 3x_{n-1} - 4x_{n-3} = 9?
\]

**Question 2.** How many solutions in integers are there to the system
\[
\begin{align*}
x_1, x_2, \ldots, x_k &\geq 1 \\
x_1 + x_2 + \ldots + x_k &= m
\end{align*}
\]
Prove your answer. How many subsets of \( \{1, 2, \ldots, n\} \) contain no subset of the form \( \{i, i+1\}, 1 \leq i \leq n-1 \)?

**Question 3.** State and prove Burnside’s lemma. [You may assume without proof that if a group \( G \) acts on a set \( X \) then \( |\text{Stab}(x)| \cdot |\text{Orb}(x)| = |G| \), where \( \text{Stab}(x) \) is the stabilizer of \( x \) and \( \text{Orb}(x) \) is its orbit.] How many essentially different ways are there to paint the corners of a solid cube with 5 colours?

**Question 4.** State and prove the Principle of Inclusion-Exclusion. The chromatic function of a graph \( G \) is the function \( \chi_G : \mathbb{N} \to \mathbb{N} \) given by
\[
\chi_G(k) = |\{ c : V(G) \to \{1, 2, \ldots, k\} : c \text{ is a proper vertex colouring} \}|.
\]
Prove using the Principle of Inclusion/Exclusion that the chromatic function is in fact a polynomial in \( k \). Show, moreover, that if \( G \) has \( n \) vertices and \( m \) edges then the leading terms of \( \chi_G \) are
\[
\chi_G(k) = k^n - mk^{n-1} + \ldots
\]

**Question 5.** Let \( b, k, n, r \) be positive integers satisfying
\[
bk = nr \quad k < n \quad b \leq \binom{n}{k}.
\]
Prove that there is a family \( \mathcal{F} \) of \( k \)-subsets of \( \{1, 2, \ldots, n\} \) with \( |\mathcal{F}| = b \) such that for all \( x \in \{1, 2, \ldots, n\} \) the number of sets in \( \mathcal{F} \) containing \( x \) is \( r \).

**Question 6.** Suppose that \( G \) is a bipartite graph with bipartition \( (X, Y) \). Define the deficiency of a subset \( S \subset X \) to be \( \text{def}(S) = |S| - |N(S)| \). Prove that the maximum size of a matching in \( G \) is
\[
|X| - \max \{ \text{def}(S) : S \subseteq X \}.
\]

**Question 7.** Let \( G \) be a graph containing no induced subgraph isomorphic to \( P_4 \). Prove that given any ordering of the vertices of \( G \), the greedy algorithm colours \( G \) in \( \omega(G) \) colours. [Hint: If the greedy algorithm uses \( k \) colours, consider the smallest \( i \) such that \( G \) contains a clique consisting of vertices coloured \( i, i+1, i+2 \ldots, k \).]

**Question 8.** State and prove Euler’s Formula concerning planar graphs. The girth of a graph \( G \) is the length of the shortest cycle in \( G \). Prove that if \( G \) is planar and bridgeless with \( n \) vertices, \( m \) edges, and girth \( g \), then
\[
m \leq \frac{g}{g-2} (n-2).
\]

**Question 9.** Let \( G \) be a graph with vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \). The adjacency matrix of \( G \) is the \( n \times n \) matrix \( A = [a_{ij}] \) with
\[
a_{ij} = \begin{cases} 1 & v_i \text{ is adjacent to } v_j \text{ in } G \\ 0 & \text{otherwise} \end{cases}
\]
Prove that the \( ij \)-th entry of \( A^k \) is the number of walks of length \( k \) in \( G \) from \( v_i \) to \( v_j \).

**Question 10.** Let \( G \) be a connected bipartite graph which is \( k \)-regular for some \( k \geq 2 \). Prove that \( G \) is bridgeless.