There are 10 questions, of which you should attempt 8. Each question carries equal weight. Standard results may be quoted provided they are clearly stated.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A graph is \( r \)-regular if every vertex has degree \( r \). We write \( \chi(G) \) for the chromatic number of \( G \) (the minimum number of colours required to properly colour the vertices of \( G \)) and \( \chi'(G) \) for the chromatic index of \( G \) (the minimum number of colours required to properly colour the edges of \( G \)). The union of two graphs \((V_1, E_1)\) and \((V_2, E_2)\) is the graph \((V_1 \cup V_2, E_1 \cup E_2)\).

**Question 1.** How many solutions in integers are there to the system

\[
\begin{align*}
x_1, x_2, \ldots, x_k & \geq 1 \\
x_1 + x_2 + \ldots + x_k &= m
\end{align*}
\]

Prove your answer. How many subsets of \( \{1, 2, \ldots, n\} \) contain no subset of the form \( \{i, i+1\} \), \( 1 \leq i \leq n-1 \)?

**Question 2.** Let \( S \) be a set of points in the plane such that \( |x - y| \geq 1 \) for all \( x, y \in S, x \neq y \). If \( |S| = n \), show that at most \( 3n - 6 \) pairs of points in \( S \) have \( |x - y| = 1 \).

**Question 3.** Find the general solution to the following recurrence

\[
a_n + 3a_{n-1} - 10a_{n-2} = 2^n.
\]

**Question 4.** Suppose that \( A = (A_i)^n \) is a family of subsets of \( \{1, 2, \ldots, n\} \). Define an \( n \times n \) matrix \( M \) by

\[
M_{ij} = \begin{cases} 
1 & \text{if } i \in A_j \\
0 & \text{if } i \notin A_j
\end{cases}
\]

Show that if \( M \) is invertible then the family \( A \) has a system of distinct representatives.

**Question 5.** Prove that if \( G \) is a 3-regular graph with a Hamilton cycle then \( \chi'(G) = 3 \).

**Question 6.** State and prove Burnside’s lemma. [You may assume without proof that if a group \( G \) acts on a set \( X \) then \( |\text{Stab}(x)| \cdot |\text{Orb}(x)| = |G| \), where \( \text{Stab}(x) \) is the stabilizer of \( x \) and \( \text{Orb}(x) \) is its orbit.] How many different ways are there to colour a five by five square grid with 2 colours if rotations (but not reflections) count as the same colouring?

**Question 7.** Suppose that \( n = 2^p + 1 \) for some integer \( p \). Prove, by considering the chromatic number or otherwise, that \( K_n \) cannot be written as the union of \( p \) bipartite graphs.

**Question 8.** State and prove the Principle of Inclusion-Exclusion. Use it to prove that if we define

\[
\phi(n) = |\{i : 1 \leq i \leq n, (i, n) = 1\}|
\]

then if \( p_1, p_2, \ldots, p_k \) are the prime divisors of \( n \) we have

\[
\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right).
\]

**Question 9.** Prove that if \( G \) is a graph with minimum degree at least 2 then \( G \) contains a cycle. Deduce that every tree (a connected acyclic graph) contains a vertex of degree exactly 1. Prove that a graph with \( n \) vertices and \( e \) edges is a tree if and only if it is acyclic and \( e = n - 1 \).

**Question 10.** Prove that a graph on \( 2n \) vertices which does no contain a triangle has at most \( n^2 \) edges.