There are 10 questions, of which you should attempt 8. Each question carries equal weight.

All graphs we consider are simple – that is they have no loops or multiple edges. An independent set in a graph is a subset of the vertices, not two of which are joined by an edge. A graph is \(k\)-partite if its vertex set can be partitioned into \(k\) independent sets.

**Question 1.** Prove that if \(G\) is a graph with \(e(G) \geq n(G)\) then \(G\) contains a cycle. Now let \(A_1, A_2, \ldots, A_n\) be \(n\) distinct subsets of \(\{1, 2, \ldots, n\}\). Prove that there exists some \(i \in \{1, 2, \ldots, n\}\) such that the sets \(A_j \setminus \{i\}, j = 1, 2, \ldots, n\) are all distinct.

**Question 2.** State and prove Hall’s Theorem concerning matchings in bipartite graphs. [No result essentially equivalent to Hall’s theorem may be assumed.] Let \(G\) be a bipartite graph with bipartition \(\{X, Y\}\) and suppose that \(d(x) = k\) for all \(x \in X\) and \(d(y) = l\) for all \(y \in Y\). Prove that \(G\) contains a matching of the smaller of \(X\) and \(Y\) into the larger.

**Question 3.** Solve the following recurrence relation:

\[
x_n - x_{n-1} - 5x_{n-2} - 3x_{n-3} = 0
\]

\[
x_0 = 1, \quad x_1 = 2, \quad x_3 = 19
\]

**Question 4.** State and prove an explicit formula for the number of integer solutions to the equation \(\sum_{i=1}^{r} x_i = n\) subject to \(1 \leq x_i \leq a_i\).

**Question 5.** Prove that if \(G\) is a graph with \(n\) vertices and maximum degree \(\Delta\) then \(G\) contains an independent set of size at least \(n/(\Delta + 1)\).

**Question 6.** The Catalan numbers, \((C_i)_{i=1}^{\infty}\), satisfy the recurrence

\[
C_1 = 1, \quad C_n = \sum_{i=1}^{n-1} C_i C_{n-i}.
\]

Let \(F(t) = \sum_{i=1}^{\infty} C_i t^i\) be the ordinary power series generating function of the sequence of Catalan numbers. Prove that \(F(t) = t + (F(t))^2\), and hence (or otherwise) derive explicit expressions for \(F(t)\) and \(C_n\).

**Question 7.** State and prove Burnside’s lemma. [You may assume without proof that if a group \(G\) acts on a set \(X\) then the product, for any element \(x \in X\), of the sizes of the stabilizer and the orbit of \(x\) is \(|G|.|\).] How many differently coloured octahedra can be made from balls and sticks if the balls available come in three different colours? [Hint: The group of rotational symmetries of the octahedron has 24 elements.]
Question 8. Suppose $G$ is a graph with $n(G) = 2m + 1$ such that $G$ does not contain a 3-cycle. Prove that $e(G) \leq m^2 + m$.

Question 9. Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ with the property that each $v_i$ is adjacent to at most $k - 1$ of the vertices $v_1, v_2, \ldots, v_{i-1}$. Show that $G$ is $k$-partite.

Question 10. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets of a $v$-set $X$, each of size $k$. Assume that each pair $\{x_1, x_2\} \subseteq X$ is contained in exactly $\lambda > 0$ of the $B_i$. Prove that if $\lambda < k < v$ then $b \geq v$. 