WORK EXACTLY 8 PROBLEMS.

1. If graph $G$ has cycles, then the length of the shortest cycle in $G$ is called the girth of $G$.
   (a) Prove that if $G$ is a connected planar graph with $p$ vertices, $q$ edges, and girth $k$, then $q \leq (p-2)(\frac{k}{k-2})$.
   (b) Use (a) to prove that $K(3,3)$ is nonplanar.
   (c) Use (a) to prove that the Petersen graph (the unique $(3,5)$-cage) is nonplanar.

2. Let $G$ be a simple graph with $p$ vertices, $q$ edges, and $k$ components. Prove:
   (a) $q \geq (p-k)$, with equality if and only if every component of $G$ is a tree.
   (b) $G$ has at least $q - p + k$ distinct cycles. (Cycles are distinct if and only if they have different edge sets).

3. (a) Show that in any graph $G$, a vertex of odd degree must always be connected by a path to another vertex of odd degree.
   (b) Prove: If a simple graph $G$ has two vertices that are not connected by a path of length 3 or less, then every pair of vertices in $G$ are connected by a path of length 3 or less.

4. (a) State P. Hall’s theorem on the existence of a system of distinct representatives.
   (b) Using (a) (or an equivalent theorem) prove the following: If $G$ is a finite bipartite graph which is regular of positive degree, then $G$ has a perfect matching.

5. (a) Define: $G$ is a tournament graph.
   (b) Prove that every tournament contains a Hamiltonian path.

6. Let $S$ be a finite set of objects and let $A_i$ be the subset of $S$ consisting of those objects possessing property $P_i$ for $1 \leq i \leq m$.
   (a) Prove the following form of the inclusion-exclusion principle: The number of objects of $S$ which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by $|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m+1}|A_1 \cap A_2 \cap \cdots \cap A_m|$, where each sum is taken over the appropriate possible subsets of indices.
   (b) Use (a) to prove: If $n$ is a positive integer whose prime factorization is $n = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k}$, then the number of integers from 1 to $n$ (inclusive) that are relatively prime to $n$ is the function $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k})$.

7. (a) Establish a recurrence relation for $f(n)$, where $f(n)$ is the number of regions created by $n$ mutually intersecting planes in 3-dimensional space such that every three planes meet in one point, but no four planes have a point in common.
   (b) Establish a recurrence relation for $g(n)$, where $g(n)$ is the number of ways to pair off with nonintersecting lines $2n$ different points on a circle.

8. Solve the following recurrence relation: $f(n) = 3f(n-2) - 2f(n-3)$ for $n \geq 3$ where $f(0) = 2, f(1) = -2, f(2) = 21$. (Note the order of this recurrence).

9. Let $B = \{B_1, B_2, \ldots, B_b\}$, $|B_i| = k$ for $1 \leq i \leq b$, be a family of subsets (called blocks) of a $v$-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}$,
1 \leq j < m \leq v, occurs in exactly \lambda > 0 blocks.
(a) Prove that each element of \( X \) occurs the same number of times \( r \) among the blocks.
(b) Prove that if \( k < v \), then \( b \geq v \). (Hint: Use an incidence matrix and determinant).

10. Prove the formula that counts the number of solutions \((x_1, x_2, \ldots, x_k)\), with \( x_i \) nonnegative integers, \( 1 \leq i \leq k \), to \( x_1 + x_2 + \cdots + x_k = r \).