WORK EXACTLY 8 PROBLEMS. Each question carries equal weight

All graphs we consider are simple - that is they have no loops or multiple edges. A bridge in a connected graph $G$ is an edge $e$ of $G$ such that $G - e$ is disconnected. We write $\chi(G)$ for the vertex chromatic number of $G$; the minimum number of colors with which we can properly color the vertices of $G$.

1. Determine the number of ways of selecting $r$ distinct integers out of the first $n$ positive integers such that the selection does not include 2 consecutive integers.

2. Let $D_r$ be the number of derangements of an $r$-set. Use a combinatorial argument to establish the identity $\sum_{r=0}^{n} \frac{D_r}{r!(n-r)!} = 1$.

3. Prove that if $G = (V, E)$ is a simple graph with $|V| = 2m$, and $G$ has no 3-cycles, then $|E| \leq m^2$.

4. (i) Show that every graph contains two vertices of equal degree.
   (ii) Determine all graphs which have exactly one pair of vertices of equal degree.

5. Let $G$ be a connected bipartite graph which is $k$-regular for some $k \geq 2$. Prove that $G$ is bridgeless.

6. (i) Show that a $k$-chromatic graph can be oriented in such a way that a longest directed path has at most $k$ vertices.
   (ii) Suppose $G$ can be oriented in such a way that no directed path contains more than $k$ vertices and suppose further that $G$ has no directed cycles. Prove that $\chi(G) \leq k$.

7. Consider the recurrence relation $a_n = 3a_{n-1} - 4a_{n-3}$, for $n \geq 3$, where $a_0 = 8, a_1 = (-1), a_2 = (-9)$. Determine an explicit formula for $a_n$.

8. (i) Prove that if $G$ is a graph in which the degree of every vertex is at least two, then $G$ contains a cycle.
   (ii) A tree is a connected acyclic graph. Use (i) to establish that a nontrivial tree contains a vertex of degree one.
   (iii) Prove that a graph $G$ with $p$ vertices and $q$ edges is a tree if and only if $G$ is acyclic and $q = p - 1$.

9. Let $B = \{B_1, B_2, \ldots, B_k\}$ be a family of subsets (called blocks) of a $v$-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}, 1 \leq j < m \leq v$, occurs in exactly $\lambda > 0$ blocks. Prove that if $\lambda < |B_i| < v$, for $1 \leq i \leq v$, then $b \geq v$. (Hint: Use an incidence matrix and determinant.)

10. Let $G$ be a permutation group acting on a set $X$. The stabiliser of a point $x \in X$ is $G_x = \{g \in G : gx = x\}$ and the orbit of $x$ is $O_x = \{gx : g \in G\}$.
    (i) State and prove Burnside’s Lemma.
    (ii) A cube with edge length 2 is constructed by gluing together 8 cubes with edge length 1. The smaller cubes come in three colors. How many different cubes with edge length 2 can be constructed? (Hint: The group of rotational symmetries of a cube has 24 permutations.)