1. Find the Cauchy function for the differential equation \((\frac{1}{t^6} x')' + \frac{12}{t^8} x = 0\) and use your answer to solve the initial value problem (IVP)

\[
(\frac{1}{t^6} x')' + \frac{12}{t^8} x = \frac{2}{t^6}, \quad x(1) = 1, \quad x'(1) = 2.
\]

Also find the Green’s function for the boundary value problem

\[
(\frac{1}{t^6} x')' + \frac{12}{t^8} x = \frac{2}{t^6}, \quad x(1) = 0, \quad x(2) = 0.
\]

2. State and prove the variation of constants formula for the vector IVP

\[
x' = A(t)x + b(t), \quad x(t_0) = x_0.
\]

3. Using Putzer’s algorithm and the variation of constants formula solve the IVP

\[
x' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.
\]

4. (a) Use the eigenpair method to find two linearly independent solutions of the vector differential equation

\[
x' = \begin{bmatrix} -1 & 6 \\ 1 & 4 \end{bmatrix} x.
\]

(b) Prove that your solutions in (a) are linearly independent.

(c) Use (b) to find a fundamental matrix.

(d) Use (c) to find \(e^{At}\), where

\[
A := \begin{bmatrix} -1 & 6 \\ 1 & 4 \end{bmatrix}.
\]

5. Prove that \(\| \cdot \|_1\) is the matrix norm corresponding to the vector traffic norm (\(l_1\) norm), which is defined by

\[
\|x\|_1 = \left\| \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \right\|_1 = \max_{1 \leq k \leq n} |x_k|,
\]

then \(\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|\), where \(A\) is an \(n \times n\) constant matrix.

6. Consider the self-adjoint DE \((p(t)x')' + q(t)x = 0\), where we assume \(p(t) > 0\) on \([a, b]\) and \(p\) and \(q\) are continuous on \([a, b]\). Prove that no nontrivial solution of this DE has an infinite number of zeros in \([a, b]\).