1. (a) Use Putzer’s algorithm to find $e^{At}$ given that $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

(b) Given $B$ is a $2 \times 2$ constant matrix such that $e^{Bt} = \begin{bmatrix} (1-t)e^{2t} & t e^{2t} \\ -te^{2t} & (1+t)e^{2t} \end{bmatrix}$, solve the initial value problem

$$x' = Bx + \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2. Assume $p(t)$, $q(t)$, and $r(t)$ are continuous on the half open interval $[a, b)$, with $p(t) > 0$, $r(t) > 0$ on $[a, b)$. Show that if $\lim_{t \to b^-} p(t) = 0$, then eigenfunctions corresponding to distinct eigenvalues for the singular Sturm-Liouville problem

$$Lx = (p(t)x')' + q(t)x = -\lambda r(t)x,$$

$$\alpha x(a) - \beta x'(a) = 0, \quad \alpha^2 + \beta^2 > 0$$

$x(t)$, $x'(t)$ are bounded on $[a, b)$, are orthogonal with respect to the weight function $r(t)$. As a completely separate problem find a constant $a$ so that $x_1(t) = t + a$, $x_2(t) = e^t$ are orthogonal with respect to the weight function $r(t) = e^{2t}$ on $[0, 1]$.

3. (a) Find the minimum value of

$$Q[x] = \int_1^2 \left\{ t(x'(t))^2 + 4t x^2(t) \right\} \, dt$$

subject to $x(1) = 1$, $x(2) = 4$. Verify that this problem does have a global minimum.

(b) Show that the problem

$$I[x] = \int_0^{2\pi} \left[ e^{2t} x^2(t) - 3e^t x(t)x'(t) - (\sin t)(x'(t))^2 \right] \, dt$$

subject to $x(0) = 2$, $x(3) = 1$, has no local extremums.

4. (a) Find the Floquet multipliers for the system

$$x' = \begin{bmatrix} -2 & \cos t \\ 0 & -2 + \cos t \end{bmatrix} x.$$

By just looking at the Floquet multipliers that you found, what stability conclusion can you draw.

(b) In general, prove that if the Floquet system $x' = A(t)x$ with minimum positive period of $A(t)$ is $\omega$ and if $\mu$ is a Floquet multiplier, then there is a nontrivial solution $x(t)$ satisfying $x(t + \omega) = \mu x(t)$ for $-\infty < t < \infty$. 
5. By finding an appropriate Green’s function solve the boundary value problem

\[ (e^{2t}x')' = e^{3t}, \quad x(0) = 0, \quad x(\log(2)) = 0. \]

Using your answer, solve the boundary value problem

\[ (e^{2t}x')' = e^{3t}, \quad x(0) = 3, \quad x(\log(2)) = 0. \]

6. Write the van der Pol equation \( x'' + \mu(x^2 - 1)x' + x = 0 \) as a 2-dimensional system in the standard way. Determine the stability of the trivial solution for all values of the real parameter \( \mu \).

7. Apply the LaSalle Invariance Theorem to the system

\[
\begin{align*}
x' &= -y \\
y' &= -yx^2 + 4x^3.
\end{align*}
\]

(Hint: \( V(x, y) = ax^4 + y^2 \)).

8. Determine if each of the following has a nontrivial periodic solution or not

(a) \[
\begin{align*}
x' &= x + y - x \left( \frac{3}{2}x^2 + y^2 \right) \\
y' &= -x + y - y \left( \frac{3}{2}x^2 + y^2 \right)
\end{align*}
\]

(b) \[
\begin{align*}
x' &= y \\
y' &= -x - y + x^2 + 2y^2.
\end{align*}
\]