(1) Define \( \alpha : [-1, 1] \to \mathbb{R} \) by
\[
\alpha(x) := \begin{cases} 
-1, & x \in [-1, 0]; \\
1, & x \in (0, 1].
\end{cases}
\]
Let \( f : [-1, 1] \to \mathbb{R} \) be a function that is uniformly bounded on \([-1, 1]\) and continuous at \( x = 0 \), but not necessarily continuous for \( x \neq 0 \). Prove that \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) over \([-1, 1]\) and that
\[
\int_{-1}^{1} f(x) \, d\alpha(x) = 2f(0).
\]

(2) (a) State the Weierstrass approximation theorem.
(b) Let \([a, b] \subset \mathbb{R}\) be given. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous and that
\[
\int_{a}^{b} f(x)x^n \, dx = 0 \quad \text{for each } n = 0, 1, 2, 3, \ldots.
\]
Prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

(3) Suppose that \( \{f_n\}_{n=1}^{\infty} \) is a sequence of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) with the following property: there are numbers \( 0 < \lambda \leq \Lambda \) such that for all integers \( n > 0 \)
\[
\lambda x^n \leq f_n(x) \leq \Lambda x^n \quad \text{for each } x \in [0, 1].
\]
(a) Prove that the series \( \sum_{n=1}^{\infty} f_n(x)(1-x) \) converges pointwise but not uniformly on \([0, 1]\).
(b) Prove that the series \( \sum_{n=1}^{\infty} (-1)^n f_n(x)(1-x) \) converges uniformly on \([0, 1]\).

(4) Fix \( x_0 \in \mathbb{R} \). Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is infinitely differentiable; i.e. has continuous derivatives on \( \mathbb{R} \) of all orders. Let \( k \in \{0, 1, 2, \ldots\} \) be given.
(a) Provide the formula for \( P_k : \mathbb{R} \to \mathbb{R} \), the \( k \)-th order Taylor polynomial for \( f \) centered at \( x_0 \).
(b) Suppose that \( Q : \mathbb{R} \to \mathbb{R} \) is a polynomial of degree \( k \) satisfying
\[
\lim_{x \to x_0} \frac{f(x) - Q(x)}{(x-x_0)^k} = 0.
\]
Prove that \( Q = P_k \).

(5) Let \((X, \rho)\) be a metric space.
(a) Suppose that \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) are sequences from \( X \) such that \( \lim_{n \to \infty} x_n = x_0 \) and \( \lim_{n \to \infty} y_n = y_0 \) for some \( x_0, y_0 \in X \). Argue that
\[
\lim_{n \to \infty} \rho(x_n, y_n) = \rho(x_0, y_0).
\]
(For this part do not use, without proof, the fact that \( \rho \) is continuous.)
(b) Under the assumption that \((X, \rho)\) is compact, verify that there exist \( a, b \in X \) such that
\[
\rho(a, b) = \sup \{\rho(x, y) : x, y \in X\}.
\]
(For this part you may assume, without proof, that \( \rho \) is continuous.)

(6) (a) Let \( \{a_n\}_{n=1}^{\infty} \) be a bounded sequence from \( \mathbb{R} \). State the definitions of \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n \).
(b) Let \( \{a_n\}_{n=1}^{\infty} \) be an enumeration of the rational numbers in \((0, 1)\). Compute, with justification, the numerical values of \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n \).