Instructions: Work 5 out of the 6 questions. Each question is worth 20 points. Write on one side of the paper only. You can (and should) quote appropriate theorems as long as doing so does not trivialize the problem.

1. (a) Prove, from the definition, that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous.

(b) Prove, using the open covering definition of compactness, that the set $\{0, 1\} \cup \left\{ \frac{1}{n}, \frac{n-1}{n} : n = 1, 2, \ldots \right\}$ is compact.

2. (a) Define what it means for $f : [0, 1] \to \mathbb{R}$ to be Riemann integrable on $[0, 1]$.

(b) Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous then the function $F : [0, 1] \to \mathbb{R}$ defined by

$$F(t) = \int_0^t f(x) \, dx,$$

is differentiable, and has $F'(t) = f(t)$ for all $t \in (0, 1)$.

(c) Show that the result of the previous part does not hold for all Riemann integrable $f : [0, 1] \to \mathbb{R}$.

3. The parts of this question are unconnected.

(a) Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies $f(-1) < 0 < f(1)$. Prove directly that there exists $c \in (-1, 1)$ such that $f(c) = 0$. No form of the Intermediate Value Theorem may be used without proof.

(b) Let $(a_n)_{n \geq 1}$ be a sequence of non-increasing, non-negative, continuous functions, $a_n : \mathbb{R} \to \mathbb{R}$. Prove that if $\sum_{n=1}^{\infty} a_n(x)$ converges for all $x$, then the function

$$g(x) = \sum_{n=1}^{\infty} a_n(x)$$

is continuous.

4. The parts of this question are unconnected.

(a) Determine the values of $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{x^n}{1 + n|x|^n}$ converges, justifying your answer carefully.

(b) Let $f : \mathbb{R} \to (0, \infty)$ be non-decreasing and suppose that $\liminf_{n \to \infty} (f(n+1) - f(n)) > 0$. Prove that

$$\limsup_{x \to \infty} \frac{f(x)}{x} > 0.$$
5. (a) An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

\[ f^{(3)}(x) = f(x). \]  

(*)

Prove that there exists $M = M(R)$ such that for all $x$ with $|x| \leq R$ and all $j \geq 0$ we have

\[ |f^{(j)}(x)| \leq M. \]

(b) Suppose now that in addition to $f$ satisfying (*) we have

\[ f(0) = 1, f'(0) = f''(0) = 0. \]

Using Taylor’s theorem, or otherwise, prove that

\[ f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}. \]

6. Recall that Dini’s Theorem states the following. If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$, $f_n : K \to \mathbb{R}$ are continuous, with $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$, and in addition $f_n(x) \to f(x)$ pointwise on $K$ then $f_n \to f$ uniformly on $K$.

(a) Prove Dini’s Theorem.

(b) Give an example to show that the compactness of $K$ is a necessary condition for Dini’s Theorem. (You don’t need to have a proof of (a) to do (b).)