(1) Let \( f(x) = \frac{x^2}{1-x^2}, \ x \in (0, 1) \).
   a) By using the \( \epsilon-\delta \) definition of the limit only, prove that \( f \) is continuous on \((0, 1)\).
   b) Is \( f \) uniformly continuous on \((0, 1)\)? Prove your answer.

(2) Let \((X, d)\) and \((Y, \rho)\) be metric spaces.
   a) Prove: If \((X, d)\) is an unbounded and connected metric space, then for every \( x_0 \in X \) and every \( r > 0 \) the set \( \{ x \in X : d(x, x_0) = r \} \) is nonempty.
   b) Let \( f, g : X \to Y \) be continuous functions, and \( E \) is dense subset of \( X \). Prove that \( f(E) \) is dense in \( Y = f(X) \); and if \( g(x) = f(x) \) for all \( x \in E \), then \( g(x) = f(x) \) for all \( x \in X \).

(3) a) Let \( \{a_n\}_{n=1}^\infty \) be a sequence of of positive real numbers. Prove that \( \sum_{n=1}^\infty a_n < \infty \) implies that \( \sum_{n=1}^\infty \sqrt{a_na_{n+1}} < \infty \) and that the converse is false.
   b) Let \( \{x_n\}_{n=1}^\infty \) and \( \{y_n\}_{n=1}^\infty \) be bounded sequences in \( \mathbb{R} \). Prove that
   \[ \lim_{n \to \infty} (x_n + y_n) \geq \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n, \]
   and show that the inequality can be strict.

(4) a) Let \( \{f_n\}_{n=1}^\infty \) be a sequence of real-valued continuous functions such that \( f_n \to f \) uniformly on \([0, 1]\). Prove that \( \lim_{n \to \infty} f_n(x_n) = f(x) \); whenever \( x_n, x \in [0, 1] \) satisfying \( x_n \to x \), as \( n \to \infty \).
   b) Prove that the series: \( x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} + \cdots \) converges uniformly on \([a, \infty)\) for every \( a > 0 \); but not uniformly on \([0, b]\) for any \( b > 0 \).

(5) a) Let \( f \) be a bounded real-valued function on \([-1, 1]\) and \( \alpha(x) = 0 \) if \( x \leq 0 \), \( \alpha(x) = 1 \) if \( x > 0 \). Prove that:
   \( f \in \mathcal{R}(\alpha)[-1, 1] \) (i.e., \( f \) is Riemann integrable with respect to \( \alpha \) on \([-1, 1]\)) if and only if \( f \) is right continuous at \( x = 0 \).
   b) Let \( \phi \) be a real-valued function defined on \([0, 1]\) such that \( \phi, \phi' \) and \( \phi'' \) are continuous on \([0, 1]\). Prove that
   \[ \int_0^1 \cos x \frac{x\phi'(x) - \phi(x) + \phi(0)}{x^2} \, dx < \frac{3}{2} \|\phi''\|_\infty, \]
   where \( \|\phi''\|_\infty := \sup_{x \in [0, 1]} |\phi''(x)| \). (Note, the constant \( \frac{3}{2} \) in the inequality may not be the smallest possible constant).

(6) Let \( \{f_n\}_{n=1}^\infty \) be a sequence of real-valued continuous functions defined on \([0, 1]\) such that \( \int_0^1 |f_n(y)| \, dy \leq 3 \), for all \( n \in \mathbb{N} \). Define \( g_n : [0, 1] \to \mathbb{R} \) by:
   \[ g_n(x) = \int_0^1 \sqrt{x + y} \, f_n(y) \, dy. \]
   Prove that \( \{g_n\}_{n=1}^\infty \) contains a subsequence that converges uniformly on \([0, 1]\).

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Masters & Ph.D. Qualifying Exam

Analysis: Math 825/826

June 1, 2009, 2:00-6:00 p.m., Avery Hall 110

- Work 5 complete question out of 6.
- Each problem is worth 20 points.
- Write on one side of the paper only and hand your work in order.
- Do not interpret a problem in such a way that it becomes trivial.