(1) Let \( f(x) = \frac{x}{1-x}, \ x \in (0,1). \)

a) (10 points) By using the \( \varepsilon-\delta \) definition of the limit only, prove that \( \lim_{x \to t} f(x) = f(t) \), for every \( t \in (0,1) \), i.e., \( f \) is continuous on \( (0,1) \). (Note: You are not allowed to trivialize the problem by using properties of limits).

b) (10 points) Is \( f \) uniformly continuous on \( (0,1) \)? Justify your answer.

(2) Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be sequences in a metric space \( (X,d) \). Assume that \( \lim_{n \to \infty} x_n = x_0 \) and \( \lim_{n \to \infty} y_n = y_0 \) for some \( x_0, y_0 \in X \).

a) (10 points) Prove that \( \lim_{n \to \infty} d(x_n, y_n) = d(x_0, y_0) \).

b) (10 points) Is the set \( E := \{x_n: n = 0, 1, 2, \cdots \} \) compact in \( X \)? Prove your answer.

(3) Let \( f: (a,b) \to \mathbb{R} \) be a function.

a) (14 points) Prove the following: If \( f' \) exists and is bounded on \( (a,b) \), then \( \lim_{x \to b^+} f(x) \) exists.

b) (6 points) Prove or disprove: If \( f' \) exists on \( (a,b) \) and \( \lim_{x \to b^+} f(x) \) exists, then \( f' \) is bounded on \( (a,b) \).

(4) Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be sequences of real numbers.

a) (10 points) Prove that: \( \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \); whenever the right hand side of the inequality is not of the form \( \infty - \infty \).

b) (10 points) Assume \( \beta > 0, \ a_n > 0, \ n = 1, 2, 3, \cdots \) and the series \( \sum_{n=1}^{\infty} a_n \) is divergent. Prove that the series \( \sum_{n=1}^{\infty} \frac{a_n}{\beta + a_n} \) is divergent.

(5) Let \( f_n(x) = \frac{x}{1+nx^2} \), for \( n \in \mathbb{N} \). Let \( C[a,b] \) denote the metric space of real-valued continuous function on \( [a,b] \) with metric \( \rho \) defined by: \( \rho(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)| \) for \( f, g \in C[a,b] \).

Let \( \mathcal{F} := \{f_n : n = 1, 2, 3, \cdots \} \) and \([a,b]\) be any compact segment in \( \mathbb{R} \).

a) (6 points) Is \( \mathcal{F} \) equicontinuous on \( [a,b] \)? Justify your answer.

b) (6 points) Is \( \mathcal{F} \) compact in the metric space \( (C[a,b], \rho) \)? Justify your answer.

c) (8 points) Let \( p > 0 \) be fixed. Prove that, for any \( a > 0 \), the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} f_n(x) \) converges uniformly on \( [a,\infty) \).

(6) Let \( \alpha(x) = 0 \) if \( x \leq 0; \ \alpha(x) = 1 \) if \( x > 0, \) and \( \beta(x) = x. \) Assume that \( f \) is a bounded real-valued function on \( [-1,1] \).

a) (10 points) Prove the following: \( f \in \mathcal{R}(\alpha) \) on \( [-1,1] \) if and only if \( \lim_{x \to 0^+} f(x) = f(0) \).

In the case \( f \in \mathcal{R}(\alpha) \) on \( [-1,1] \), find the value of \( \int_{-1}^{1} f(x) d\alpha(x) \). (Note: \( f \in \mathcal{R}(\alpha) \) on \( [-1,1] \) means that \( f \) is Riemann integrable with respect to \( \alpha \) on \( [-1,1] \)).

b) (10 points) Assume that the function \( f \in \mathcal{R}(\beta) \) on \( [-1,1] \), \( m \leq f(x) \leq M \) for \( x \in [-1,1] \), \( \phi \) is continuous on \( [m,M] \), and \( g(x) := \phi(f(x)) \). Prove that \( g \in \mathcal{R}(\beta) \) on \( [-1,1] \).