**Answer 5 of the following 6 questions. All questions are of equal weight.**

1. Let $f_n(x) = (1 - x^2)x^{2n}$, for $n = 0, 1, 2, ..$ and for $x \in [-1, 1]$.

(a) Show that the series $\sum_{n=0}^{\infty} f_n(x)$ converges pointwise but not uniformly, and find the limit.

(b) Show that the series $\sum_{n=0}^{\infty}(-1)^n f_n(x)$ converges uniformly on $[-1, 1]$.

2. (a) Give an example of a continuous function $f : [1, \infty) \to [0, \infty)$ such that:

(i) $f(x) \geq 0$ for all $x \in [1, \infty)$,
(ii) the improper integral $\int_1^{\infty} f(x)dx$ converges,
(iii) the series $\sum_{n=1}^{\infty} f(n)$ diverges.

(b) Prove or disprove: The series $\sum_{n=1}^{\infty} \frac{\ln(1+1/n)}{n}$ converges.

3. (a) Let $T : C[0, 1] \to C[0, 1]$ be defined by

$$Tf(x) := x + \int_0^x tf(t)dt.$$

Show that $T$ is a contraction mapping on C[0,1]. That is, show that there exists $k$ with $0 < k < 1$ such that $\parallel Tf - Tg \parallel \leq k \parallel f - g \parallel$ where $\parallel \cdot \parallel$ denotes the max norm in $C[0, 1]$.

(b) The metric space $C[0, 1]$ is complete in the metric $d(f, g) := \sup_x |f(x) - g(x)|$. Find a closed and bounded subset of $C[0, 1]$ which is not compact.

4. Let $b_1 \in R$ be given and for $n = 1, 2, ...$ let

$$b_{n+1} := \frac{1 + b_n^2}{2}$$

Define the set

$$B := \{b_1 \in R : \lim_{n\to\infty} b_n \text{ converges}\}$$

(i) Describe (identify) the set $B$ and prove your assertion.

(ii) Find $\lim_{n\to\infty} b_n$ for each $b_1 \in B$.

5. (a) Prove that if $f$ is continuous, nonnegative, but not identically zero on $[0, 1]$ and if $g$ is strictly increasing on $[0, 1]$, then the Riemann-Stieltjes integral

$$\int_0^1 f dg > 0$$

(b) Let $\{r_1 = 0, r_2 = 4, r_3, r_4, ...\}$ be an enumeration of the rational numbers $Q$ in $[0, 4]$. Let

$$h(x) = \begin{cases} 1/n, & \text{if } x = r_n,\ n = 1, 2, ... \\ 0, & \text{if } x \in [0, 4] \cap (R \setminus Q) \end{cases}$$

(i) Prove or disprove: $h$ is of bounded variation on $[0, 4]$.

(ii) Could $h$ be of bounded variation on $[0, 1]$? Why or why not?

6. (a) Suppose that $\{f_n\}$ is a sequence of continuous real-valued functions that converges uniformly on a compact set $D \subset R$. Show that $\{f_n\}$ is uniformly bounded and equicontinuous on $D$.

(b) Let $S$ be a metric space with metric $d$ and let $H$ and $K$ be disjoint closed and nonempty subsets of $S$. If $K$ is compact, show that

$$\inf\{d(x, y) : x \in H, y \in K\} > 0.$$