Answer five out of the following seven questions.
If you answer more than five, make clear which questions you want graded.
All questions carry equal weight.

1. Suppose \( g_n(x) \) is a sequence of real-valued functions defined on a set \( S \subseteq \mathbb{R} \). Suppose further that \( 0 \leq g_{n+1}(x) \leq g_n(x) \) for all \( n \in \mathbb{N} \) and \( x \in S \) and that \( g_n(x) \to 0 \) uniformly on \( S \). Prove that \( \sum_{n=1}^{\infty} (-1)^n g_n(x) \) converges uniformly on \( S \).

2. (a) Let \( S \) be a set of real numbers. Prove carefully from the definitions that

\[
\sup \{ |x - y| : x, y \in S \} = \sup(S) - \inf(S)
\]

(b) Hence prove that whenever \( f \) is a Riemann integrable function on \( [a, b] \) then \( |f| \) is also Riemann integrable on \( [a, b] \).

3. Let \( f(x, y) \) be continuous on \( [a, b] \times [c, d] \). Prove that the function

\[
g(x) := \int_c^d f(x, y) \, dy
\]

is continuous on \( [a, b] \).

4. (a) Define the term

it compact set in a metric space.

(b) Let \( (X, \rho) \) be a metric space and let \( (x_n) \) be a sequence in \( X \) that converges to \( a \in X \). Prove directly from the definition of compactness that

\[
K := \{a\} \cup \{x_n : n \in \mathbb{N}\}
\]

is compact.

5. Let \( h : \mathbb{R} \to \mathbb{R} \). Show that \( \lim_{t \to a^+} h(t) \) exists if and only if, given any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for all \( a < x, y < a + \delta \).

6. Suppose that \( f : [0, +\infty) \to \mathbb{R} \) and that

\[
f(0) = 0, f'(0) = 1 \quad \text{and} \quad f''(x) \leq 0 \quad \text{for all} \quad x > 0
\]

(a) Prove that \( f(x) \leq x \) for all \( x \geq 0 \).

(b) Prove that \( f(x)/x \) is decreasing on \( [0, +\infty) \).

7. Suppose that a subset \( S \) of a metric space has the property that given any \( a, b \in S \), there is a continuous function \( \gamma : [0, 1] \to S \) such that \( \gamma(0) = a \) and \( \gamma(1) = b \). Prove that \( S \) is connected.