(1) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function on \( \mathbb{R} \). Fix \( c \in \mathbb{R} \), and suppose that \( f \) has the following property: there is an \( L \) such that for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\left| \frac{f(r) - f(c)}{r - c} - L \right| < \varepsilon \quad \text{whenever} \quad r \in \mathbb{Q} \quad \text{and} \quad 0 < |r - c| < \delta.
\]
Prove that \( f \) is differentiable at \( c \) and that \( f'(c) = L \).

(2) (a) Carefully state the Mean Value Theorem.
(b) Let \( \lambda > 0 \) be given. Show that there is no function \( f : \mathbb{R} \to \mathbb{R} \) that is differentiable at every \( x \in \mathbb{R} \) and such that
\[
f'(x) = \begin{cases} 
0, & x < 0 \\
\lambda, & x \geq 0.
\end{cases}
\]

(3) Define \( \alpha, \beta, f : \mathbb{R} \to \mathbb{R} \) by
\[
\alpha(x) := \begin{cases} 
-1, & x < 0 \\
4, & x \geq 0;
\end{cases} \quad \beta(x) := \begin{cases} 
-1, & x \leq 0 \\
4, & x > 0;
\end{cases} \quad \text{and} \quad f(x) := \begin{cases} 
0, & x < 0 \\
1, & x \geq 0;
\end{cases}
\]
(a) Determine whether \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) over \([-1, 1]\). If it is, evaluate \( \int_{-1}^{1} f \, d\alpha \).
(b) Determine whether \( f \) is Riemann-Stieltjes integrable with respect to \( \beta \) over \([-1, 1]\). If it is, evaluate \( \int_{-1}^{1} f \, d\beta \).

(4) Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of \( \mathbb{R} \)-valued functions defined on \( \mathbb{R} \). Suppose that for each \( n \in \mathbb{N} \) and each \( x \in \mathbb{R} \) we have \( 0 \leq f_{n+1}(x) \leq f_n(x) \) and that \( f_n \to 0 \) uniformly on \( \mathbb{R} \). Prove that \( \sum_{n=1}^{\infty}(-1)^nf_n(x) \) converges uniformly on \( \mathbb{R} \).

(5) Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces.
(a) Carefully state the definition of continuity for a mapping \( f : X \to Y \).
(b) Carefully state the definition of compactness for \( X \).
(c) Suppose that \( X \) is compact and that \( f : X \to Y \) is continuous. Prove that \( f(X) \) must be a compact subset of \( Y \).

(6) Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a convergent sequence of real numbers. Suppose that \( M \in \mathbb{R} \) is such that both \( \lim_{n \to \infty} a_n \neq M \) and \( a_n \neq M \) for all \( n \in \mathbb{N} \). Show that there must be a \( d > 0 \) such that \( |a_n - M| > d \) for all \( n \in \mathbb{N} \).