Masters & Ph.D. Qualifying Exam
Analysis: Math 825/826
January 22, 2009, 2:00-6:00 p.m., Avery Hall 347

• Work 5 complete question out of 6. • Each problem is worth 20 points.
  • Write on one side of the paper only and hand your work in order. • Do not interpret a problem in such a way that it becomes trivial.

(1) a) (10 points) Let \( a_1 = 1 \) and \( a_{n+1} = \sqrt{a_n + 1} \) for \( n \in \mathbb{N} \). Show that \( \lim_{n \to \infty} a_n \) exists, and find its value.

b) (10 points) For which real numbers \( x \) does the series \( \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x} \) converge? Justify your answer.

(2) Let \( f : (0,1] \to \mathbb{R} \) be a uniformly continuous function on \((0,1]\).

a) (10 points) State the definition of the uniform continuity of \( f \) on \((0,1]\), and use it to show that \( g(x) = \ln x \) is not uniformly continuous in \((0,1]\).

b) (10 points) Prove that \( f \) can be uniquely extended to a continuous function on \([0,1]\).

(3) (20 points) Let \( \{x_n\}_{n=1}^{\infty} \) be a bounded sequence in \( \mathbb{R} \). Define \( A \) by:

\[
A := \{a \in \mathbb{R} : \text{the set } \{n \in \mathbb{N} : x_n < a\} \text{ is a finite set}\}.
\]

Prove that:

\[
\sup A = \lim_{n \to \infty} x_n.
\]

(4) This problem has two independent parts.

a) (10 points) Consider the metric space \((\mathbb{Q}, d)\) where \( \mathbb{Q} \) denotes the rational numbers and \( d(x, y) = |x - y| \). Let \( E := \{x \in \mathbb{Q} : x > 0, 2 < x^2 < 3\} \). Is \( E \) closed and bounded in \( \mathbb{Q} \)? Is \( E \) compact in \( \mathbb{Q} \)? Justify your answers.

b) (10 points) Let \( f \) be a continuous real-valued function on \([0,1]\). Prove that there exists at least one point \( \xi \in [0,1] \) such that

\[
\int_{1}^{0} x^4 f(x) \, dx = \frac{1}{5} f(\xi).
\]

(5) This problem has two independent parts.

a) (10 points) Let \( f \) be a function defined on \([0,1]\) by: \( f(x) = 0 \) if \( x \) is irrational or \( x = 0 \), and \( f(x) = 1/q \) if \( x = p/q \) is rational where \( p, q \) have no common factor. Find the values of the upper and lower Riemann integrals, \( \int_0^1 f \, dx \), \( \int_0^1 f \, dx \). Is \( f \) Riemann integrable on \([0,1]\)?

b) (10 points) Let \( \{g_n\}_{n=1}^{\infty} \) be a sequence of real-valued, continuously differentiable functions on \([0,1]\), such that, for all \( n \in \mathbb{N} \),

\[
|g_n'(x)| \leq \frac{1}{\sqrt{x}}, \quad 0 < x \leq 1; \quad \text{and} \quad \int_0^1 g_n(x) \, dx = 0.
\]

Prove that the sequence \( \{g_n\}_{n=1}^{\infty} \) has a subsequence that converges uniformly on \([0,1]\).

(6) This problem has two independent parts.

a) (10 points) Let \( f_n(x) = \frac{nx}{1+n(1+x^2)} \), \( n \in \mathbb{N} \). Does the sequence \( \{f_n\}_{n=1}^{\infty} \) converge uniformly on \( \mathbb{R} \)? and to what?

b) (10 points) Let \( g \) be a real-valued continuous function on \([0,1]\). Evaluate the following limit:

\[
\lim_{n \to \infty} n \int_0^1 x^n g(x) \, dx. \quad \text{(Hint: First, consider: } \lim_{n \to \infty} n \int_0^1 x^n (g(x) - g(1)) \, dx).\]