Analysis Qualifier, January 2007

Answer 5 of the following 6 questions. All questions are of equal weight. Note that in problems 2, 4 and 6, part (a) is independent of part (b).

1. Let \( f(x) = \int_1^x \frac{1}{t} \, dt \) for \( x > 0 \).

   (a) Use an \( \epsilon-\delta \) proof to show that \( f \) is continuous on \((0, \infty)\).

   (b) Use an \( \epsilon-\delta \) proof to show that \( f \) is differentiable on \((0, \infty)\).

2. (a) Let \( f : [1, +\infty) \rightarrow [0, +\infty) \) be piecewise continuous and nonincreasing. Show that the series \( \sum_{n=1}^{\infty} f(n) \) converges if and only if the improper integral \( \int_1^{\infty} f(x) \, dx \) converges.

   (b) Let \( g(t) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{t^j}{n^2} \).

      i. For what real values of \( t \) does this series converge absolutely?

      ii. For what real values of \( t \) does this series converge conditionally?

3. (a) Let \( B \) be a compact subset of \( \mathbb{R} \) and let \( f : B \rightarrow \mathbb{R} \) be continuous. Show that there exists a point \( b \in B \) such that \( f(x) \leq f(b) \) for all \( x \in B \) (i.e. \( f \) attains its maximum value).

   (b) Let \( f \) be a positive continuous function defined on \( \mathbb{R} \) such that \( \lim_{|x| \rightarrow \infty} f(x) = 0 \). Show that \( f \) attains its maximum value.

4. (a) Let \( f : [0, 1] \rightarrow [0, 1] \) be defined by

\[
    f(t) = \begin{cases} 
    0 & \text{if } x \text{ is irrational} \\
    1 & \text{if } x = 0 \\
    \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in reduced form, where } p \text{ and } q \text{ are positive integers}
    \end{cases}
\]

   Prove that \( f \) is Riemann integrable on \([0, 1]\) and that \( \int_0^1 f(x) \, dx = 0 \).

   (b) For \( n = 1, 2, 3, \ldots \) and \( x \in [0, 1] \), let

\[
    g_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}.
\]

   Prove or disprove: The family of functions \( \mathcal{G} = \{ g_n : n = 1, 2, 3, \ldots \} \) is equicontinuous on \([0, 1]\).
5. Let $E$ be a compact subset of $\mathbb{R}$.

(a) Let $g : E \to \mathbb{R}$ be a continuous function such that $g(x) \neq 0$ for all $x \in E$. Show that there exists $c > 0$ such that $|g(x)| \geq c$ for all $x \in E$.

(b) Suppose that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $E$ to a bounded function $f$, and that a sequence of functions $\{g_n\}_{n=1}^{\infty}$ converges uniformly to a continuous function $g$, where $g(x) \neq 0$ for all $x \in E$. Prove that the sequence of functions $\{f_n/g_n\}_{n=1}^{\infty}$ is defined everywhere on $E$ for large $n$ and converges uniformly on $E$ to $f/g$.

6. (a) Let $f$ be a function of bounded variation on $[a, b]$. Furthermore, assume that for some $c > 0$, $|f(x)| \geq c$ on $[a, b]$. Show that $g(x) = 1/f(x)$ is of bounded variation on $[a, b]$.

(b) Prove that every open set $A \subseteq \mathbb{R}$ can be written as a finite or countable union of disjoint open intervals $(a_j, b_j)$, where at most one $a_j = -\infty$ and at most one $b_j = \infty$. 