Answer 5 of the following 7 questions. All have equal weight.

1. Let the power series \( \sum_{n=0}^{\infty} a_n x^n \) and \( \sum_{n=0}^{\infty} b_n x^n \) have radii of convergence \( R_1 \) and \( R_2 \) respectively.
   
   (a) If \( R_1 \neq R_2 \), prove that the radius of convergence, \( R \), of the power series \( \sum_{n=0}^{\infty} (a_n + b_n) x^n \) is \( \min\{R_1, R_2\} \). What can be said about \( R \) when \( R_1 = R_2 \)?
   
   (b) Prove that the radius of convergence, \( R \), of the power series \( \sum_{n=0}^{\infty} a_n b_n x^n \) satisfies \( R \geq R_1 R_2 \). Show by means of an example that the inequality can be strict.

2. (a) Give a careful \( \epsilon - \delta \) proof that \( g(x) = \sqrt{x} \) is continuous on \([0, \infty)\).
   
   (b) Assume that \( f \) is differentiable at \( a \). Evaluate
   
   \[
   \lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a} \quad (n \in \mathbb{N})
   \]

3. (a) Let \( S \) be the set of all sequences \((q_1, q_2, \ldots)\) of rational numbers which converge to zero. Is \( S \) countable or uncountable?
   
   (b) Let \((X, d)\) be a compact metric space and let \( f : X \to X \) be continuous and onto. Let \( g : X \to X \) and suppose \( g \circ f \) is continuous. Prove that \( g \) must be continuous.

4. (a) Let
   
   \[
   f_n(x) = \begin{cases} 
   \frac{1}{n} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \\
   0 & \text{otherwise}
   \end{cases}
   \]
   
   Show that \( \sum_{n=1}^{\infty} f_n \) does not satisfy the Weierstrass \( M \)-Test but that it nevertheless converges uniformly on \( \mathbb{R} \).
   
   (b) Suppose that \( f \) is continuous and \( f(x) \geq 0 \) on \([0, 1]\). If \( f(0) > 0 \), prove that \( \int_{0}^{1} f(x) \, dx > 0 \).

5. Let \( a_{m,n} \geq 0 \) \((m, n \in \mathbb{N})\) and suppose that the partial sums
   
   \[
   \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn}
   \]
   
   are bounded above. Prove carefully that \( \sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} a_{mn}) \) and \( \sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} a_{mn}) \) exist and are equal.

6. Suppose \( f' \) exists and is increasing on \((0, \infty)\) and that \( f \) is continuous on \([0, \infty)\) with \( f(0) = 0 \). Show that \( g(x) = f(x)/x \) is increasing on \((0, \infty)\).

7. (a) Let \( f \) be continuous on \([0, 1]\) and \( f(0) = f(1) = 0 \). Show that there is a sequence of polynomials \( p_n \) such that \( x(1-x)p_n(x) \) converges to \( f \) uniformly.
   
   (b) Let \( f \) be uniformly continuous on a bounded set \( E \subseteq \mathbb{R} \) and let \( a \in E \). Prove that \( \lim_{x \to a} f(x) \) exists.