(1) (a) Let \( \{a_k\} \) be a sequence of real numbers such that the series \( \sum_{k=1}^{\infty} a_k \) is convergent and that the series \( \sum_{k=1}^{\infty} a_k^2 \) is divergent. Prove that the series \( \sum_{k=1}^{\infty} a_k \) does not converge absolutely.

(b) Consider the series \( \sum_{k=1}^{\infty} \frac{1}{1+z^k} \), where \( z \in \mathbb{C} \). Prove that the series diverges for all \( |z| \leq 1 \); and the series converges absolutely for all \( |z| > 1 \).

(2) Let \((X, d)\) be a compact metric space and \( f: X \to \mathbb{R} \) is a continuous function on \( X \).

(a) Prove that \( f(X) \) is compact in \( \mathbb{R} \).

(b) Prove that there exists a point \( a \in X \) such that \( f(a) = \sup f(X) \).

(3) (a) Assume that \( \sum_{k=1}^{\infty} a_k \) is a convergent series of nonnegative real numbers. Prove that the series \( \sum_{k=1}^{\infty} a_k^x \) converges uniformly on \([1, \infty)\).

(b) Prove: the series \( \sum_{k=0}^{\infty} \frac{x^3}{(1+x^3)^k} \) converges uniformly on \([a, b]\) for every \( 0 < a < b \); but the convergence is not uniform on \([0, b]\) for any \( b > 0 \).

(4) Let \( f \) be given by: \( f(x) = \sum_{n=1}^{\infty} \frac{|x|}{x^2 + n^2} \).

(a) Show that \( f \) is well defined on \( \mathbb{R} \).

(b) Prove that \( f \) is continuous on \( \mathbb{R} \), but \( f \) is not differentiable at 0.
(5) (a) Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\). Prove that there exists \( c \in (a, b) \) such that
\[
\frac{1}{b - a} \int_a^b f(x) \, dx = f(c).
\]
(b) Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function on \([0, 1]\) with \( f(0) = 0 \). Prove that there exists a sequence of polynomials \( \{Q_n\} \) such that \( xQ_n(x) \to f(x) \) uniformly on \([0, 1]\).

(6) (a) Let \( \alpha \) be monotone increasing function on \([a, b]\) and assume that \( \alpha \) is continuous at some point \( s \in [a, b] \). Prove that if \( f(s) = 1 \) and \( f(x) = 0 \) for \( x \neq s \), then \( f \in R(\alpha)[a, b] \) (i.e., \( f \) is Riemann integrable with respect to \( \alpha \) on \([a, b]\)) and that
\[
\int_a^b f \, d\alpha = 0.
\]
(b) Let \( \alpha_n \) be a sequence of monotone increasing functions on \([a, b]\) and assume that \( f \) is a bounded function on \([a, b]\) such that \( f \in R(\alpha_n)[a, b] \) for every \( n \in \mathbb{N} \). Prove that: if \( \lim_{n \to \infty} \alpha_n(x) = 0 \) for each \( x \in [a, b] \), then \( \int_a^b f \, d\alpha_n \to 0 \), as \( n \to \infty \).

(7) (a) Let \( f_n : [a, b] \to \mathbb{R}, n = 1, 2, \ldots \) be a uniformly bounded sequence on \([a, b]\), and assume that \( f_n \in R[a, b] \). Let \( F_n(x) = \int_a^x f_n(t) \, dt, x \in [a, b] \). Prove that there exists a subsequence \( \{F_{n_k}\} \) of \( \{F_n\} \) such that \( \{F_{n_k}\} \) converges uniformly on \([a, b]\).
(b) Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function on \([a, b]\). Call \( f \) uniformly differentiable on \([a, b]\) if, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \), such that
\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon
\]
for all \( x, t \in [a, b] \) with \( 0 < |t - x| < \delta \). Prove that if \( f' \) is continuous on \([a, b]\) then \( f \) is uniformly differentiable on \([a, b]\).