Section I. Groups

1. Recall that the centralizer of a subgroup $H$ in a group $G$ is
   
   \[ C_G(H) = \{ g \in G \mid gh = hg \text{ for all } h \in H \}. \]

   (a) Prove that if $H$ is normal in $G$, then $C_G(H)$ is normal in $G$.

   (b) Prove that if $H$ is normal in $G$, then $G/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$ (the group of automorphisms of $H$).

2. (a) Suppose $H$ is a subgroup of a group $G$ and $[G : H] = 7$. Prove $G$ contains a normal subgroup $N$ such that $N \subset H$ and $[G : N] \leq 7!$.

   (b) Prove $7!$ is the best possible bound for the previous part — i.e., prove there is a group $G$ and a subgroup $H$ with $[G : H] = 7$ such that for every normal subgroup $N$ of $G$ with $N \subset H$, we have $[G : N] \geq 7!$.

3. Suppose $G$ is a simple group of order $168 = 2^3 \cdot 3 \cdot 7$. (Yes, there is such a group.)

   (a) How many elements of order 7 does $G$ have?

   (b) Show that $G$ has at least 14 elements of order 3.

Section II. Linear Algebra and Modules

4. Let $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$, with entries in $\mathbb{C}$.

   (a) Find the Jordan canonical form of $A$.

   (b) Let $B = \begin{bmatrix} 0 & -9 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{bmatrix}$, with entries in $\mathbb{C}$. Is $A$ similar to $B$?

5. Let $R$ be a ring with identity and let $M$ be a left $R$-module. Recall that the annihilator of $M$ in $R$ is
   
   \[ \text{ann}_R(M) = \{ r \in R \mid rm = 0 \text{ for all } m \in M \}. \]

   (a) Prove that $\text{ann}_R(M)$ is a 2-sided ideal of $R$.

   (b) Suppose $M$ is an abelian group (i.e., a $\mathbb{Z}$-module) such that $|M| = 400$ and $\text{ann}_{\mathbb{Z}}(M)$ is the ideal generated by 20. How many possibilities, up to isomorphism, are there for $M$?

6. Let $F$ be a field and $V$ a vector space (not necessarily finite-dimensional) over $F$. Prove that every linearly independent subset of $V$ is contained in a basis for $V$. 
Section III. Rings, Fields and Galois Theory

7. Let \( R \) be an integral domain with field of fractions \( Q \). Let \( P \) be a prime ideal of \( R \) and let

\[
S = \left\{ \frac{r}{d} \in Q \mid d \notin P \right\}.
\]

(a) Show that \( S \) is a subring of \( Q \).
(b) Show that

\[
I = \left\{ \frac{p}{d} \mid p \in P, d \notin P \right\}
\]

is a prime ideal of \( S \).

8. Prove \( \mathbb{Z}[2i] = \{ a + 2bi \mid a \text{ and } b \text{ are integers} \} \) is not a PID. \textit{Hint:} One method is to use (with proof) the fact that \( 2 + 2i \) is irreducible in this ring.

9. Consider \( f(x) = x^6 + 3 \in \mathbb{Q}[x] \).

(a) Let \( \alpha \) be a root of \( f(x) \) and prove \( \mathbb{Q}(\alpha) \) is Galois over \( \mathbb{Q} \). (Hint: First show \( \alpha^3 + 1 \) is a primitive 6-th root of unity.)
(b) Find the Galois group of \( \mathbb{Q}(\alpha)/\mathbb{Q} \).