Math 817–818 Qualifying Exams and Masters Comprehensive Exam
June 6, 2003  1:00–4:00PM

• Do two of the four given problems from each of the three sections, for a total of six problems. Be sure to make it clear which six problems you want graded.

• If you have doubts about the wording of a problem or which results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

• Be sure to show your reasoning clearly and explain everything carefully.

I Groups and Geometry

I.1 Recall that the $3 \times 3$ orthogonal group, $O_3$, is defined as

$$O_3 = \{ A \in GL_3(\mathbb{R}) \mid A^T A = I_3 \}.$$  

Prove $O_3$ is an internal direct product of two non-trivial subgroups, $H$ and $K$ (i.e., prove that for some pair of non-trivial subgroups $H$ and $K$ of $G$, there is an isomorphism $H \times K \cong G$ given by $(h, k) \mapsto hk$).

I.2 Let $M$ denote the group of rigid motions of the plane and let $G$ be a finite subgroup of $M$. Prove $G$ fixes a point — that is, show there exists a point $P$ in the plane such that $g(P) = P$ for all $P$. (You are not allowed to use the theorem that classifies all finite subgroups of $M$.)

I.3 Let $G$ be a non-zero subgroup of the group of real numbers under addition $(\mathbb{R}, +)$. Prove $G$ is discrete if and only if $G$ is an infinite cyclic group. (Recall that $G$ is discrete if there is a real number $\epsilon > 0$ such that for all non-zero elements $0 \neq g \in G$, we have $|g| > \epsilon$. Here, $|g|$ denotes the absolute value of $g$, not its order.)

I.4 Let $N$ be a normal subgroup of a finite group $G$. Assume $|N| = p$, where $p$ is prime, and that $p$ is the smallest prime divisor of $|G|$. Prove $N$ is contained in the center of $G$. Hint: Consider the action of $G$ on $N$ via conjugation.

(Turn the page for sections II and III.)
II Linear Algebra

II.1 An $n \times n$ matrix $A$ is called unipotent if $A = I_n + B$ for some nilpotent matrix $B$. (A matrix $B$ is nilpotent if $B^k = 0$ for some $k \geq 1$.) Prove that if $A$ is unipotent, then $A$ is similar to a lower triangular matrix with 1’s along the diagonal:

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
* & \cdots & * & 1 & 0 \\
* & \cdots & * & * & 1
\end{bmatrix}
$$

II.2 Let $A$ be an $n \times n$ matrix with complex entries.

(a) Prove that if $A^k = I_n$ for some $k \geq 1$, then $A$ is diagonalizable.

(b) Show by example that even if the matrix $A$ in part (a) has real entries, it need not be diagonalizable over the reals.

II.3 Find the rational canonical form of the matrix with complex entries

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
z & 1 & 0 & 1
\end{bmatrix}
$$

where $z$ is an arbitrary complex number. (Your answer may or may not depend on the value of $z$.)

I.4 Recall that, if $A$ is an $n \times n$ complex matrix, then $A$ is Hermitian if $A = A^*$ and $A$ is unitary if $A^*A = I_n$. (Here, $A^*$ denotes the conjugate transpose of $A$.) Assume $A$ is Hermitian. Prove that $A$ is unitary if and only if all the eigenvalues of $A$ are $\pm 1$.

III Rings, Modules, and Fields

III.1 Let $f(x), g(x) \in \mathbb{Q}[x]$ be irreducible polynomials, and let $\alpha \in \mathbb{C}$ be a root of $f(x)$ and let $\beta \in \mathbb{C}$ be a root of $g(x)$. Prove that $f(x)$ is irreducible over $\mathbb{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbb{Q}(\alpha)$.

III.2 Let $R = \mathbb{Z}[\sqrt{-10}]$.

(a) Using that $(2 + \sqrt{-10})(2 - \sqrt{-10}) = 14$, prove $R$ is not a UFD.

(b) Prove $I = (7, 2 + \sqrt{-10})$ is a maximal ideal of $R$ that is not principal.

III.3 Prove $f(x) = 25x^5 - 6x^4 - x^2 + 5x - 16 \in \mathbb{Q}[x]$ is irreducible. Hint: Modulo 3, we have $f(x) = (x^2 + 1)(x^3 - x - 1)$.

III.4 If $M$ is a $\mathbb{Z}$-module (i.e., an abelian group), the annihilator of $M$ is defined to be

$$
ann(M) = \{n \in \mathbb{Z} \mid nx = 0, \text{for all } x \in M\}.
$$

In general, $ann(M)$ is an ideal of $\mathbb{Z}$. (You need not prove this.)

Suppose $M$ is the $\mathbb{Z}$-module presented by an $n \times n$ matrix $A$ with entries in $\mathbb{Z}$ (so that $M \cong \text{coker}(A)$).

(a) Prove $\det(A) \in ann(M)$.

(b) Assume $\det(A) \neq 0$. Prove $ann(M) = (\det(A))$ if and only if $M$ is cyclic. (Here, $(\det(A))$ denotes the principal ideal of $\mathbb{Z}$ generated by $\det(A)$.)