Section I: Groups and Geometry

1. Let $G$ be a group (not necessarily finite). A subgroup $S$ of $G$ is said to be characteristic provided $\sigma(S) \subseteq S$ for every $\sigma \in \text{Aut}(G)$.
   (a) Prove that every characteristic subgroup of $G$ is a normal subgroup of $G$.
   (b) Prove that the center of a group $G$ is a characteristic subgroup of $G$.
   (c) Prove that if $S$ is a characteristic subgroup of $G$ then $\sigma(S) = S$ for every $\sigma \in \text{Aut}(G)$.
   (d) Let $p$ be a prime and let $P$ be the subgroup of $G$ generated by all elements of $G$ whose order is a power of $p$. Prove that $P$ is a characteristic subgroup of $G$.

2. Group actions
   (a) Let $G$ be a group of order 15 acting on a set with 7 elements. Prove that there exists at least one fixed point.
   (b) Give an example of an action of $C_{15}$ on a set with 8 elements with no fixed points. Justify.

3. Rigid Motions of the Plane: Let $\tau$ be a translation of the plane, and let $\rho$ be a non-trivial rotation about some point in the plane.
   (a) Prove that $\rho \tau$ has a fixed point. ($\tau$ acts first.)
   (b) Prove that $\tau \rho$ has a fixed point.
   Give direct and rigorous proofs, without assuming the classification of symmetries of the plane.

4. Dense subgroups of $\mathbb{R}^+$.
   (a) Let $G$ be a subgroup of the additive group $\mathbb{R}$ containing arbitrarily small positive real numbers. Prove that $G$ is dense in $\mathbb{R}$. (That is, given real numbers $a$ and $b$ with $a < b$, prove that there is an element $g \in G$ with $a < g < b$.)
   (b) Prove that $G := \mathbb{Z} + \mathbb{Z}\sqrt{2}$, the additive subgroup of $\mathbb{R}$ generated by 1 and $\sqrt{2}$, is dense in $\mathbb{R}$. (One approach is to show that $G$ does not have a smallest positive element and to deduce (a) from this fact. Other methods are possible.)
Section II: Rings and Fields

5. Field Extensions
(a) Give an example of a field extension $L/F$ of finite degree having two distinct intermediate fields $K_1$ and $K_2$ such that $K_1$ and $K_2$ are isomorphic as fields. Justify.
(b) Suppose $K_1$ and $K_2$ are two distinct subfields of $\mathbb{C}$ such that $[K_1 : \mathbb{Q}] = [K_2 : \mathbb{Q}] = 2$. Prove that $K_1$ and $K_2$ are not isomorphic fields. (You may assume that every quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}(\sqrt{d})$ for some square-free integer $d \neq 1$.)

6. Prove that the polynomial $f(x) := x^5 + 6x^4 + 3x^2 + x + 2$ is irreducible over $\mathbb{Q}$. (You might find it helpful to reduce modulo 2.)

7. Let $K/k$ be a field extension (not necessarily algebraic).
(a) Let $R$ be a ring with $k \subseteq R \subseteq K$, and assume that every element of $R$ is algebraic over $k$. Prove that $R$ is a field.
(b) Let $E$ and $F$ be intermediate fields, both of them algebraic over $k$. Prove that

$$\{ \sum_{i=1}^{n} e_i f_i \mid e_i \in E, f_i \in F, n \in \mathbb{N} \}$$

is the smallest subfield of $K$ containing both $E$ and $F$.

8. Let $R$ be an integral domain (commutative with identity). Suppose that the polynomial ring $R[x]$ is a principal ideal domain. Prove that $R$ is a field.
Section III: Linear Algebra

9. Similarity
(a) How many similarity classes of $6 \times 6$ matrices with characteristic polynomial $(x^2 + 1)^3$ are there over $\mathbb{C}$? Explain.
(b) How many similarity classes of $6 \times 6$ matrices with characteristic polynomial $(x^2 + 1)^3$ are there over $\mathbb{Q}$? Justify your answer, and give one representative from each class.

10. Recall that the cokernel of an $m \times n$ matrix $\alpha$ over $\mathbb{Z}$ is the abelian group $\mathbb{Z}^m/C$, where $C$ is the subgroup of $\mathbb{Z}^m$ generated by the columns of $\alpha$. For each of the following, reduce the matrix to diagonal form by doing integer row and column operations, and express the cokernel as a direct sum of cyclic groups.

(a) $\alpha = \begin{bmatrix} 6 & 4 & 2 \\ 6 & 2 & 4 \end{bmatrix}$.

(b) $\beta = \begin{bmatrix} 6 & 6 \\ 4 & 2 \\ 2 & 4 \end{bmatrix}$.

11. Let $A$ be an $n \times n$ real matrix. Let $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$ denote the usual norm and inner product on $\mathbb{R}^n$. (Thus, viewing elements of $\mathbb{R}^n$ as column vectors, we have $\langle v, w \rangle = v^t w$ and $\|v\| = \sqrt{\langle v, v \rangle}$.) Prove that the following conditions (various criteria for $A$ to be orthogonal) are equivalent:
(a) $A^t A = I_n$.
(b) $\|A v\| = \|v\|$ for each $v \in \mathbb{R}^n$.
(c) $\langle A v, A w \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^n$.
(d) The columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.

12. Let $A = \begin{bmatrix} -1 & -9 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 2 & 7 & 2 & 0 \\ 4 & 13 & 0 & 2 \end{bmatrix}$.

(a) Show that the characteristic polynomial of $A$ is $(\lambda - 2)^4$.
(b) Find the Jordan canonical form $J$ of $A$. Justify.
(c) We know there is an invertible matrix $P$ such that $P A P^{-1} = J$. Find either $P$ or $P^{-1}$. (It’s your choice, but be sure to clarify which one you are finding!)