Master's Comprehensive and Ph.D. Qualifying Exam
Algebra: Math 817-818, January 30, 1999

Do 6 problems, 2 from each of the three sections. If you work on more than six problems, or on more than 2 from any section, clearly indicate which you want graded. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section I: Groups

[I.1] Let $S$ be the set of all positive integers $n$ such that every group of order 75 contains an element of order $n$. Determine $S$ and justify your answer.

[I.2] Let $G$ be a group of order $(p \cdot q)^2$, where $p$ and $q$ are primes with $p > 3$ and $q = p + 2$. Prove that $G$ is either cyclic or isomorphic to the product of two cyclic groups.

[I.3] Let $G$ be a finite group with $x \in G$ such that $|G| = nm$.
(a) Suppose $x \in G$ has order $n$ and let $\sigma_x \in S_G$ be the permutation such that $\sigma_x(g) = xg$ for every $g \in G$. Show that $\sigma_x$ is a product of $m$ disjoint $n$-cycles.
(b) If $n = 2$ and $m$ is odd, show that there is a homomorphism $f : G \to S_G$ such that $f(G)$ contains an odd permutation.
(c) If $n = 2$ and $m$ is odd, conclude that $G$ has a subgroup of index 2.

[I.4] Let $G$ be a group. Recall that the commutator subgroup of $G$ is defined to be the subgroup $G'$ of $G$ generated by the set of all elements of the form $x^{-1}y^{-1}xy$, where $x$ and $y$ are elements of $G$.
(a) Show that $G'$ is a normal subgroup of $G$.
(b) Show that $G/G'$ is abelian.
(c) For any normal subgroup $N$ of $G$, show that $G/N$ is abelian if and only if $G' \leq N$.

Section II: Rings and Fields

[II.5] Prove that $3x^5 + 5x^4 + 9x^3 + 6x^2 + x + 7$ is irreducible in $\mathbb{Q}[x]$, where $\mathbb{Q}$ is the field of rational numbers.

[II.6] Let $R$ be the ring of all continuous real-valued functions on the unit interval. Let $c \in [0, 1]$ and denote \{ $f \in R : f(c) = 0$ \} by $M_c$.
(a) Show that $M_c$ is a maximal ideal of $R$.
(b) Show that every maximal ideal of $R$ is of the form $M_c$ for some $c \in [0, 1]$. Hint: Use compactness.

[II.7] Let $A$ be a finite commutative ring with $1 \neq 0$. Show that every prime ideal of $A$ is maximal.

[II.8] Prove that the group of invertible elements of a finite field is cyclic.

Section III: Modules and Vector Spaces

[III.9] Let $\mathbb{C}$ be the field of complex numbers and let $x$ be an indeterminate. Let $L : \mathbb{C}[x]/(x^2(x-1)^3) \to \mathbb{C}[x]/(x^2(x-1)^3)$ be the linear transformation given by multiplying by $x$.
(a) Find the Jordan Canonical Form for $L$.
(b) Find a basis of $\mathbb{C}[x]/(x^2(x-1)^3)$ with respect to which the matrix for $L$ is the Jordan Canonical Form.

[III.10] Let $V$ be a vector space. Do not assume that $V$ is finite dimensional. Let $W \subset V$ be a spanning set for $V$. Show that $W$ contains a basis for $V$.

[III.11] Let $V$ be a finite dimensional vector space. Let $L : V \to V$ be a linear transformation. Show that there is a positive integer $n$ such that $(\ker L^n) \cap (\text{Im } L^n) = \{0\}$.

[III.12] Let $G = GL(2, \mathbb{Q})$ be the group of invertible $2 \times 2$ matrices over the rational numbers. Determine up to similarity all elements of $G$ which have order exactly 4.