Math 932-33 Comprehensive Exam May, 2009 **Name** ________________.

There are six questions, all of equal weight. Answer as many of them as completely and carefully as you can. Use white paper and write on one side of the paper only.

1. Let $\mathcal{F}$ be a nonempty subset of $C[0,1]$ and assume that every sequence in $\mathcal{F}$ contains a uniformly convergent subsequence. Show that

   (a) $\mathcal{F}$ is uniformly bounded and equicontinuous.

   (b) $\phi(x) := \sup\{f(x) : f \in \mathcal{F}\}$ is continuous on $[0,1]$.

2. (a) Show that if u and v are nonnegative functions on $[a,b]$ and $C \geq 0$ is a constant such that $v(t) \leq C + \int_a^t u(s)v(s)\,ds$ for $t \in [a,b]$, then $v(t) \leq Ce^{\int_a^t u(s)\,ds}$ for $t \in [a,b]$.

   (b) Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

   $$|f(t,x)| \leq m(t)|x|$$

   for all $(t,x) \in \mathbb{R} \times \mathbb{R}$. Assume also that $m(t) \geq 0$ for all $t \in \mathbb{R}$ and that for any $t_0 \in \mathbb{R}$

   $$\lim_{T\to\infty} \int_{t_0}^{T} m(s)ds$$

   converges (finite). Show that

   $$\lim_{t\to\infty} |x(t)|$$

   exists for each solution of the differential equation $x' = f(t,x)$.

3. Let $U \subset X$ be an open nonempty bounded subset of the Banach space X and let $S, T : \overline{U} \to X$ be compact perturbations of the identity map I. Assume further that $Tx \neq 0$ and $Sx \neq 0$ for all $x \in \partial U$. Show that if the degree of S and T with respect to U are different, (i.e., $deg(T,U) \neq deg(S,U)$), then the eigenvalue problem $Tx = \lambda Sx$ has a solution for some $x \in \partial U$, and some $\lambda < 0$.

4. (a) Consider the BVP

$$(*):\ \ x'' = f(t,x,x'),\ \ x(t_1) = c_1,\ \ x(t_2) = c_2\ \ a < t_1 < t_2 \le b,$$

where $f \in C(J \times \mathbb{R}^2, \mathbb{R}),\ \ J = (a,b]$, and is strictly increasing in x for fixed (t,x'). Assume that all solutions of the differential equation $x'' = f(t,x,x')$ exist on J. Show that the BVP (*) has at most one solution for $c_1, c_2 \in \mathbb{R}$.

(b) Let $f \in C[J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n],\ \ J = [0,T]$. Let $\Delta x = x_2 - x_1,\ \Delta x' = x_2' - x_1'$, $\Delta f = f(t,x_2,x_2') - f(t,x_1,x_1')$, where $x_i,\ x_i',\ i = 1,2$ are real variables.

Assume further that

$$\Delta x \cdot \Delta f + \parallel \Delta x' \parallel^2 > 0$$

if $\Delta x \neq 0$, and $\Delta x \cdot \Delta x' = 0$.

Show that the BVP

$$x'' = f(t,x,x'),\ x(0) = x_0,\ x(T) = x_1$$

has at most one solution.

5. Let $C[0,1]$ be the space of all continuous functions $u : [0,1] \to \mathbb{R}$ and let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Let $f \in C[0,1]$ be fixed and consider the mapping $T : C[0,1] \to C[0,1]$ defined by

$$(Tu)(t) := f(t) + \int_0^1 K(t,s)u(s)ds.$$

(a) Show that T is continuous and $T(A)$ is relatively compact for any bounded set $A \subset C[0,1]$.

(b) Show that T has a fixed point in the closed ball $B := \{u \in C[0,1] : \parallel u \parallel \le 1\}$, provided $K + N \le 1$, where $K = \sup|f(t)|$, and $N = \sup\{\int_0^1 |K(t,s)|ds : 0 \le t \le 1\}$.

6. (a) Assume you are given that the boundary of the closed n-dimensional ball

$$B(0;R) := \{x \in \mathbb{R}^n : \parallel x \parallel \le R\}$$

is not a retract of $B(0;R)$. Use this to prove the Brouwer Fixed Point Theorem. That is, show that if f is a continuous mapping of $B(0;R)$ into itself, then f has a fixed point in $B(0;R)$.

(b) Let $C[0,1]$ be the space of all continuous real-valued functions $u : [0,1] \to \mathbb{R}$. Assume that $a, f \in C[0,1]$, and consider the mapping $T : C[0,1] \to C[0,1]$ defined by

$$(Tu)(t) := f(t) + \int_0^t a(s)\sin(u(s))ds.$$

Show that T has a unique fixed point $u_0 \in C[0,1]$ if $\int_0^1 |a(s)|ds < 1$.