1. Show that the integral equation

\[ x(t) = e^{2t} \sin t + \frac{1}{2} \int_0^t s x(s) \, ds \]

has a unique continuous solution on \([0, \sqrt{3}]\).

2. Show that the set of all real sequences \(x = \{x_n\}\) with \(\lim_{n \to \infty} x_n = 0\) with the sup norm \(\|x\| := \sup\{|x_n| : n = 1, 2, 3, \cdots\}\) is a Banach space.

3. State the theorem concerning lower and upper solutions of \(x'' = f(t, x)\). Reproduce the part of the proof that if \(x(t)\) is a solution of the modified BVP

\[ x'' = F(t, x), \quad x(a) = A, \quad x(b) = B, \]

then \(x(t) \geq \alpha(t)\) on \([a, b]\).

4. Let \(x(t; a, b)\) denote the solution of the IVP

\[ x' = \arctan x, \quad x(a) = b. \]

Without solving this IVP, find

\[ \frac{\partial x}{\partial b}(t, 0, 0) \text{ and } \frac{\partial x}{\partial a}(t, 0, 0). \]

5. State and prove the Ascoli-Arzela Theorem for real-valued functions of a real variable. Give three examples where if you omit one of the three assumptions in the Ascoli-Arzela Theorem, then the conclusion of this theorem is not true.

6. Define the Hilbert projective metric \(d\) and prove the triangle inequality

\[ d(x, z) \leq d(x, y) + d(y, z) \text{ for all } x, y, z \in P^0. \]

7. Define \(P\) is a cone and normal cone. State a theorem concerning equivalent statements to \(P\) is a normal cone. Prove any two implications (one way) in your theorem.

8. Prove the following theorem: Theorem. Assume \(f : [a, b] \times [0, \infty) \to [0, \infty)\) is continuous and

\[ \lim_{x \to 0^+} \frac{f(t, x)}{x} = \infty, \quad \lim_{x \to \infty} \frac{f(t, x)}{x} = 0 \]

uniformly for \(t \in [a, b]\). Then the BVP

\[ -x'' = f(t, x), \quad x(a) = 0, \quad x(b) = 0 \]

has a positive solution.