Math 932-933 Comprehensive Exam June 2005

There are ten questions all of equal weight. Answer seven of the questions and indicate which questions you are omitting. Use white paper and write on one side of the paper only.

1. Let $C[0,1]$ be the space of all continuous real-valued functions $u : [0,1] \to R$ and let $\alpha > 0$. Assume that $f,\ h \in C[0,1]$ with $h(t) \geq 0$ for all $0 \leq t \leq 1$ and that g is a real valued continuously differentiable function whose derivative is uniformly bounded on all of R; i.e., $g : R \to R$ with $|g'(x)| \leq M$ for some $M > 0$ and for all $x \in R$. Consider the mapping $T : C[0,1] \to C[0,1]$ defined by

$$(Tu)(t) := f(t) + \alpha \int_0^t h(s)g(u(s))ds, \quad t \in [0,1].$$

Let $\beta := \displaystyle\int_0^1 h(s)ds$ and assume that $0 < M\alpha\beta < 1$. Show that T has a unique fixed point $u_0 \in C[0,1]$.

2. Let $C[0,1]$ be the space of all continuous functions $u : [0,1] \to R$ and let $K : [0,1] \times [0,1] \to R$ be continuous. Let $f \in C[0,1]$ be fixed and consider the mapping $T : C[0,1] \to C[0,1]$ defined by

$$(Tu)(t) := f(t) + \int_0^1 K(t,s)u(s)ds, \quad t \in [0,1].$$

(a) Show that T is continuous and $T(A)$ is relatively compact for any bounded set $A \subset C[0,1]$.

(b) Show that T has a fixed point in the closed ball $B := \{u \in C[0,1] : \| u \| \leq 1\}$, provided $K + N \leq 1$, where $K = \sup |f(t)|$, and $N = \sup\{\int_0^1 |K(t,s)|ds : 0 \leq t \leq 1\}$. That is, verify that the closed convex bounded subset B of $C[0,1]$ satisfies $T(B) \subset B$, and so by the Schauder Theorem, T has a fixed point in B.

3. Assume $f : R \times R^n \to R^n$ is T- periodic in t; i.e., $f(t+T,x) = f(t,x)$ for all $(t,x) \in R \times R^n$. Consider the system $u' = f(t,u)$. Assume that f is continuous and is such that for every $x \in R^n$ the IVP

$$u' = f(t,u), \quad u(0) = x \tag{1}$$

has a unique solution $u(t;x)$ on $[0,\infty)$ which depends continuously on the initial value x. (This means that for fixed t the function $P_t(x) := u(t,x)$ is a continuous function of x.) Assume that there exists $M > 0$ such that for every $t \in [0,T]$ and for every $x \in R^n$ with $\| x \| = M$ we have $x \cdot f(t,x) < 0$. Show that there exists $x_0 \in R^n$ with $\| x_0 \| \leq M$ such that $u(T;x_0) = x_0$. Use this to show that the system (1) has a periodic solution. (Hint: Look at $r(t) := \|u(t,x(t))\|^2$).

4. Suppose that X is a Banach space and T is a continuous mapping of X into X which is compact on each bounded subset A of X, and suppose further that for $0 < \lambda < 1$, the set of all solutions of the equation $x = \lambda Tx$ is bounded. (That is, there exists $M > 0$, independent of λ, such that any solution of $x = \lambda Tx$ satisfies $\| x \| \leq M$). Show that the equation $x = \lambda Tx$ has a solution for $\lambda = 1$.

5.

(a) Let $J = [0, \infty)$ and suppose that $K : J \times J \times R \to R$ is continuous and increasing in its third variable. Assume further that there exist three continuous functions $u,\ v,\ f : J \to R$ such that

$$u(t) < f(t) + \int_0^t K(t, s, u(s))ds$$

and

$$v(t) \geq f(t) + \int_0^t K(t, s, v(s))ds$$

for every $t \geq 0$, and $u(0) < v(0)$. Show that $u(t) < v(t)$ for all $t \geq 0$.

(b) Consider the periodic BVP for the forced equation

$$u'' + a(t)f(u) = g(t), \quad u(0) = u(1), \quad u'(0) = u'(1),$$

where $a,\ g \in C[0, 1]$, and $f : R \to R$ is continuous and satisfies $|f(u)| \leq M$ for all $u \in R$. Assume also that $\int_0^1 |a(s)|ds > 0$. Show that a necessary condition for the existence of a solution to the above periodic BVP is

$$\frac{|\int_0^1 g(s)ds|}{\int_0^1 |a(s)|ds} \leq M.$$

6. Let $\mathcal{F}$ be a nonempty subset of $C[0, 1]$ and assume that every sequence in $\mathcal{F}$ contains a uniformly convergent subsequence. Show that

(a) $\mathcal{F}$ is uniformly bounded and equicontinuous.

(b) $\phi(x) := \sup\{f(x) : f \in \mathcal{F}\}$ is continuous on $[0, 1]$.

7. Assume that $f : R^2 \to R$ is continuous and satisfies

$$|f(t, x)| \leq m(t)|x|$$

for all $(t, x) \in R^2$. Assume also that $m(t) \geq 0$ for all $t \in R$ and that for any $t_0 \in R$

$$\lim_{T \to \infty} \int_{t_0}^T m(s)ds$$

converges (finite). Show that

$$\lim_{t \to \infty} |x(t)|$$

exists for each solution of the differential equation $x' = f(t, x)$.

8. (a) Consider the BVP

$$x'' = f(t, x, x'), \quad x(t_1) = c_1, \quad x(t_2) = c_2, \quad a < t_1 < t_2 \leq b, \tag{2}$$

where $f \in C(J \times R^2, R)$, $J = (a, b]$, and is strictly increasing in x for fixed (t, x'). Assume that all solutions of the differential equation $x'' = f(t, x, x')$ exist on J. Show that the BVP (2) has at most one solution for $c_1, c_2 \in R$.

(b) Let $f \in C[J \times R^n \times R^n, R^n]$, $J = [0, T]$. Let $\Delta x = x_2 - x_1$, $\Delta x' = x_2' - x_1'$, $\Delta f = f(t, x_2, x_2') - f(t, x_1, x_1')$, where x_i, x_i', $i = 1, 2$ are real variables.

Assume further that

$$\Delta x \cdot \Delta f + \| \Delta x' \|^2 > 0$$

if $\Delta x \neq 0$, and $\Delta x \cdot \Delta x' = 0$. Show that the BVP

$$x'' = f(t, x, x'),\ x(0) = x_0,\ x(T) = x_1$$

has at most one solution.

9. Prove that if $f : [a, b] \times R \to R$ is continuous and $\alpha(t)$, $\beta(t)$ are lower and upper solutions of $x'' = f(t, x)$ with $\alpha(t) \leq \beta(t)$ on $[a, b]$, then the BVP

$$x'' = f(t, x), \quad x(a) = A, \quad x(b) = B,$$

where $\alpha(a) \leq A \leq \beta(a)$, $\alpha(b) \leq B \leq \beta(b)$ has a solution satisfying $\alpha(t) \leq x(t) \leq \beta(t)$ on $[a, b]$. In your proof of $\alpha(t) \leq x(t) \leq \beta(t)$ on $[a, b]$ just do $\alpha(t) \leq x(t)$ on $[a, b]$.

10. Let $x(t; a, b)$ denote the solution of the IVP

$$x' = 8 - 6x + x^2, \quad x(a) = b.$$

Without solving this IVP, find $z(t) := \frac{\partial x}{\partial b}(t; 0, 2)$ and $w(t) := \frac{\partial x}{\partial a}(t; 0, 2)$. Use your answers to approximate $x(t; h, 2)$, when h is close to zero and $x(t; 0, k)$ when k is close to 2.