1. State the Picard–Lindelof theorem and then maximize the $\alpha$ in that theorem by choosing the appropriate rectangle concerning the solution of the IVP

$$x' = t + x^2, \quad x(0) = 1.$$ 

2. State and prove the contraction mapping theorem.

3. Approximate the solution of the IVP

$$x' = \frac{1}{1 + x^2}, \quad x(0) = 0$$

by finding the second Picard iterate $x_2(t)$ and find an estimate for how good an approximation it is.

4. Let $x(t;a,b)$ denote the solution of the IVP

$$x' = \arctan x, \quad x(a) = b.$$ 

Find

$$\frac{\partial x}{\partial b}(t;0,0)$$

and

$$\frac{\partial x}{\partial a}(t;0,0).$$
1(10pts) Let $\gamma$ be a periodic orbit of a differentiable dynamical system $\dot{x} = f(x), x \in \mathbb{R}^n$. Let $\Sigma$ be a plane perpendicular to $\gamma$ at a point $p \in \gamma$. If defined, let $P : \Sigma \to \Sigma$ with $x \mapsto P(x) = \phi^{\tau(x)}(x)$, where $\tau$ is the first connecting time from $\Sigma$ to $\Sigma$ following the flow $\phi$ of the equation. Prove that $P$ is a local diffeomorphism.

2(10pts) Let $\phi_t$ denote the solution operator for a continuous dynamical system $\dot{x} = f(x), x \in \mathbb{R}^n$. Let $\omega(x) = \{y \in \mathbb{R}^n : \exists t_k \to \infty \text{ s.t. } \phi^{t_k}(x) \to y \text{ as } k \to \infty\}$ be the omega-limit set. Show that $\omega(x)$ is invariant, i.e., $\phi_t(\omega(x)) = \omega(x)$ for any $t$.

3(10pts) Consider the equations with a real parameter $\mu$.
\[
\begin{cases}
\dot{x} = -x + \mu y^2 \\
\dot{y} = y - x^2
\end{cases}
\]
Approximate the parameterized stable manifold at the origin $(0,0)$ to the 2nd order.

4(10pts) Consider 2 systems of equations
\[
\begin{align*}
\dot{x} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x \\
\dot{y} &= \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} y
\end{align*}
\]
(a) Construct an explicit conjugacy $h : \mathbb{R}^2 \to \mathbb{R}^2$ between them.
(b) Explicitly verify that the mapping $h$ you have found above is in fact a conjugacy.

5(10pts) STATE and PROVE the Stable Manifold Theorem for the system of equations
\[
\begin{cases}
\dot{x} = Ax + f(x,y) \\
\dot{y} = By + g(x,y), \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m
\end{cases}
\]
if $f, g$ are uniform Lipschitzian.

6(10pts) Consider the system of equations with parameter $0 < \alpha < 1$:
\[
\begin{cases}
\dot{x} = x(1-x) - y \\
\dot{y} = (x-\alpha) + (x-\alpha)^2
\end{cases}
\]
(a) Find the parameter value $\alpha = \alpha_0$ and the corresponding equilibrium point branch $(x_\alpha, y_\alpha)$ such that the system at $\alpha = \alpha_0$ satisfies the eigenvalue condition of Hopf bifurcation.
(b) Use a sequence of transformations to change the system into the following polar coordinate form
\[
\begin{cases}
\dot{r} = \mu r + r^2 C_3(\theta, \mu) + r^3 C_4(\theta, \mu) + \ldots \\
\dot{\theta} = \beta(\mu) + r D_3(\theta, \mu) + \ldots
\end{cases}
\]
(Hint: use $\mu = \alpha - \alpha_0, x := x - x_\alpha, y := y - y_\alpha$ as the first transformation, and find TWO more transformations of your own to complete the process.)
(c) The Hopf Bifurcation Theorem states that if
\[
K := \frac{1}{2\pi} \int_0^{2\pi} \left[ C_3(\theta, 0) - \frac{1}{\beta(0)} C_3(\theta, 0) D_3(\theta, 0) \right] d\theta \neq 0
\]
then a periodic orbit must bifurcate from the equilibrium point at $\mu = 0$. Determine the stability of the periodic orbit. (You may need this trigonometric identity: $\int \cos^n \theta d\theta = \frac{1}{n} \cos^{n-1} \theta \sin \theta + \frac{n-1}{n} \int \cos^{n-2} \theta d\theta$.)
(d) Summarize your analysis with a local bifurcation diagram near $\mu = 0$. 

The End