Math 935/936–Comprehensive Examination

Thursday, June 2, 2005, 1:00-5:00p.m., Avery Hall 111

• Work problems (1) and (2) and one of the remaining problems (3), (4). • Note the point value for each problem.

• Write on one side of the paper only and hand your work in order.

(1) (40 points) Let $H = L^2[0,\pi]$ and $k(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(n+1)x \sin ny$ for $x, y \in [0,\pi]$. We say k is symmetric if $k(x,y) = \overline{k(y,x)}$ a.e. in $[0,\pi] \times [0,\pi]$. For $u \in H$ define $(Ku)(x) := \int_0^\pi k(x,y)u(y)dy$. In this case we say, $k(x,y)$ is the kernel of K.
a) Prove that $K \in \mathcal{L}(H)$ and K is compact.
b) Show that the kernel of K^* is $\overline{k(y,x)}$.
c) Find $\|K\|$.
d) Prove that $\sigma_p(K) = \emptyset$ and explain why $k(x,y)$ is not a symmetric kernel.
e) Show that $0 \in \sigma_r(K)$, the residual spectrum of K.
(comments: Avoid writing the integral sign as much as possible. Remember that $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal basis for $L^2[0,\pi]$.)

(2) (60 points) Let $\theta \in \mathbb{R} \setminus \{0\}$ and $A : \mathcal{D}(A) \subset L^2[0,1] \to L^2[0,1]$ is given by:
$Au = -u'' + \theta \langle u, 1 \rangle = -u'' + \theta \int_0^1 u(t)dt$, where the domain of A is:

$$\mathcal{D}(A) = \{u \in L^2[0,1] : u, u' \text{ are absolutely continuous on } [0,1], u'' \in L^2[0,1], \text{ and } B_1 u := u(0) = 0, B_2 u := u(1) = 0\}.$$

a) Prove that A is a closed operator. (Hint: Find the Green's function for the problem $(-(\frac{d}{dx})^2, B_1, B_2)$ on $[0,1]$ and remember the inversion theorem.)
b) Is A self-adjoint?
c) Find a necessary and sufficient condition for $\lambda = 0 \notin \sigma_p(A)$.
d) Find a formula for A^{-1} whenever it exists.
e) Here and in part f), fix $\theta = 1$. Without finding all eigen-pairs of A, does the set of all eigen-functions of A form an orthonormal basis for $L^2[0,1]$?
f) consider the boundary-value problem:

$$-u''(x) + \int_0^1 u(t)dt = \lambda \cos(u(x)), \quad 0 < x < 1, \tag{0.1}$$

$$u(0) = 0, \quad u(1) = 0, \tag{0.2}$$

where $\lambda \in \mathbb{R}$. Prove that the boundary-value problem (0.1)-(0.2) has a unique solution u provide $|\lambda| < \lambda_0$, for some $\lambda_0 > 0$.

(3) (20 points) For $m \in \mathbb{N}$, $r > 0$, and $\phi \in \mathcal{D}(\mathbb{R})$, let f_m be given by:

$$< f_m, \phi >= \int_{\mathbb{R}} \frac{\sin mx}{mx^2} \left(\phi(x) - \phi(0)\chi_{(-r,r)}(x)\right) dx,$$

where χ denotes the characteristic function.
(a) Prove that $f_m \in \mathcal{D}'(\mathbf{R})$, for every $m \in \mathbb{N}$; and show f_m is independent of r.
(b) Prove that $f_m \to 0$ in $\mathcal{D}'(\mathbb{R})$, as $m \to \infty$.

(4) (20 points) Given the Sokhotsky's formula

$$\lim_{a \to 0^+} \frac{1}{x - ia} = i\pi\delta(x) + \mathcal{P}\frac{1}{x} \text{ in } \mathcal{S}'(\mathbb{R})$$

where $\mathcal{P}\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$ is the canonical regularization of the function $\frac{1}{x}$.
a) Prove that $e^{-ax}H_0(x) \xrightarrow{\mathcal{S}'(\mathbb{R})} H_0(x)$; as $a \to 0^+$.
b) Use part a) to compute the Fourier Transform of $\mathcal{P}\frac{1}{x}$ in $\mathcal{S}'(\mathbb{R})$.