Math 935/936 Comprehensive Exam

June 5, 1998

- Given that the set $\left\{e_n(x) = \sqrt{\frac{2}{\pi}} cos(\frac{2n-1}{2}x) : n = 1,\ 2,\ 3, \cdots\right\}$ is an orthonormal basis for $X = L^2(0, \pi)$. For $0 < x, y < \pi$, let
$$k(x, y) = \sum_{n\geq 1} \frac{1}{n^2} e_{n+1}(x) e_n(y),$$
and
$$Ku(x) = \int_0^\pi k(x, y)u(y)dy.$$
(a) Show that K is a compact linear operator from X into X.
(b) Compute $\|K\|$.
(c) Prove that k is an unsymmetric kernel.
(d) Explain why the following statement is valid: if $\lambda \in \mathbf{C} \setminus \{0\}$, then $\lambda \in \rho(K)$, the resolvent set of K.
(e) Prove that $\lambda = 0 \in \sigma_r(K)$, the residual spectrum of K.

- For $m = 1,\ 2,\ 3, \cdots$ and $\phi \in \mathcal{D}(\mathbf{R})$, let f_m be given by:
$$\langle f_m, \phi\rangle = \int_{\mathbf{R}} \frac{cos(mx)}{x}(\phi(x) - \phi(0)\chi_{(-r,r)}(x))dx,$$
where $r > 0$, and χ denotes the characteristic function.
(a) By either a direct proof or by the aid of a theorem, show that $f_m \in \mathcal{D}'(\mathbf{R})$, for $m = 1,\ 2,\ 3, \cdots$.
(b) Show that f_m is independent of r.
(c) Prove that $f_m \to 0$ in $\mathcal{D}'(\mathbf{R})$, as $m \to \infty$.

- Let $Lu(x) = -(xu')' = -xu'' - u',\ B_1u = u(1)$ and $B_2u = u(e)$.
(a) Find the Green's function $g(x, y)$ for the problem (L, B_1, B_2) on the interval $(1, e)$.
(b) Prove that the only solution of the following boundary value problem:
$$Lu(x) = \lambda sin(u(x)),\ 1 < x < e,$$
$$u(1) = 0,\ u(e) = 0.$$
is the trivial solution; whenever $|\lambda| < \lambda_0$, for some $\lambda_0 > 0$. **You need not find the numerical value of λ_0.**

- Let $A : \mathcal{D}(A) \subset L^2_r(0, 1) \to L^2_r(0, 1)$ be given by (where $L^2_r(0, 1)$ denotes the usual L^2 space of **real-valued functions** on (0,1)) :
$\mathcal{D}(A) = \{u \in L^2_r(0, 1)$: u is absolutely continuous on $[\alpha, 1]$, for any $\alpha > 0$,
$xu' \in L^2_r(0, 1),\ u(1) = 0\}$,
$Au = xu' + \theta \int_0^1 u(t)dt$,
where $\theta \in \mathbf{R}$.
(a) Find all eigenvalues and the corresponding eigenfunctions for A.
(b) Find a formula for A^{-1} whenever A^{-1} exists.