1. Let \( X \) be the Hilbert space \( L^2(0,1) \) with the usual inner product. Let 
\[
Af = \frac{df}{dx}, \quad \mathcal{D}(A) = \{ f \in X \mid f \text{ is absolutely continuous, } \frac{df}{dx} \in X, \text{ and } f(1) = 0 \}.
\]
Prove that \( A \) is a closed operator.

2. For each of the operators \( A \) given below, classify each \( \lambda \in \mathbb{C} \) as being in either \( \rho(A) \) (the resolvent set of \( A \)), \( \sigma_p(A) \) (the point spectrum of \( A \)), \( \sigma_c(A) \) (the continuous spectrum of \( A \)), or \( \sigma_r(A) \) (the residual spectrum of \( A \)).

(a) Let \( X \) be the Hilbert space \( L^2(0,\infty) \) with the usual inner product. Define the operator \( A \) by 
\[
(Af)(x) = x^2 f(x), \quad \mathcal{D}(A) = \{ f \in X \mid \int_0^{\infty} |x^2 f(x)|^2 \, dx < \infty \}.
\]

(b) Let \( X \) be a Hilbert space, and let \( A \) be an orthogonal projection on \( X \). (Here’s are reminder of the definition of an orthogonal projection: Let \( M \) be a closed subspace \( X \). Every \( z \in M \) can be written uniquely as \( z = x + y \) with \( x \in M \) and \( y \in M^\perp \). Then \( Az = x \) is an orthogonal projection.)

3. Show that any weakly convergent sequence in a Banach space is a bounded sequence.

4. Let \( \ell^2 = \{(x_1, x_2, x_3, \ldots) \mid x_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |x_k|^2 < \infty \} \), with the usual inner product. Define 
\[
A(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).
\]
Find \( A^* \). Is \( A \) a normal operator?

5. Let \( X \) be a Hilbert space, and let \( A \) be the infinitesimal generator of a strongly continuous semigroup \( e^{At} \) on \( X \).

(a) Suppose that \( K \in \mathcal{B}(X) \) is a compact operator. Prove that 
\[
\lim_{t \to 0} \|(e^{At} - I)K\| = 0.
\]

(b) Show, by constructing a counterexample, that the following is not true for all infinitessimal generators \( A \):
\[
\lim_{t \to 0} \|(e^{At} - I)\| = 0.
\]
6. With $\Omega \subset \mathbb{R}^n$ a bounded open set, consider the boundary value problem

$$
\begin{align*}
-\Delta u &= f \text{ on } \Omega \\
\frac{\partial u}{\partial \nu} &= g \text{ on } \Gamma,
\end{align*}
$$

(1)

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

(a) What are the appropriate choices of a Hilbert space $V$, a bilinear form $a(\cdot, \cdot)$, and a linear functional $F(\cdot)$ for the variational formulation of this problem?

(b) Show that a solution exists only if

$$
\int_{\Omega} f d\Omega = -\int_{\Gamma} g d\Gamma.
$$

Also show that if $u$ is a solution of (1) then so is $u + c$, where constant $c$ is arbitrary.

7. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the equation

$$
-\Delta u = u^2,
$$

where $\Omega$ is a bounded, connected open set in $\mathbb{R}^n$, use the Strong Maximum Principle to prove that $u$ cannot attain its infimum on $\Omega$ unless $u$ is identically zero.

8. Recalling the Lax-Milgram Theorem, let

$$
\begin{align*}
V &= H^1_0(0,1), \\
a(u, v) &= \int_0^1 [u'v' + c(x)u'v] \, dx \text{ for } u, v \in V \\
F(v) &= \int_0^1 fv \, dx \text{ for } v \in V,
\end{align*}
$$

where $f \in L^2(0,1)$, and $|c(x)| \leq c_1$, $0 \leq x \leq 1$.

(a) Show that $a(\cdot, \cdot)$ is a bounded, unsymmetric bilinear form.

(b) Show that $a(\cdot, \cdot)$ is $V$-elliptic if $c_1$ is sufficiently small, or if $c(x)$ is a constant function.

(c) Find some $c(x)$ and $u \in H^1_0(0,1)$ such $a(u, u) < 0$.

(d) What is the boundary value problem associated with the variational problem, “find $u \in V$ such that $u$ solves $a(u, v) = F(v)$ for all $v \in V$”?