Applied Mathematics Comprehensive Exam, January 2004

1. Let X be the Hilbert space $L^2(0,1)$ with the usual inner product. Let

$$Af = \frac{df}{dx}, \qquad \mathcal{D}(A) = \{f \in X \mid f \text{ is absolutely continuous}, \ \frac{df}{dx} \in X, \text{ and } f(1) = 0\}.$$

   Prove that A is a closed operator.

2. For each of the operators A given below, classify each $\lambda \in \mathbf{C}$ as being in either $\rho(A)$ (the resolvent set of A), $\sigma_p(A)$ (the point spectrum of A), $\sigma_c(A)$ (the continuous spectrum of A), or $\sigma_r(A)$ (the residual spectrum of A).

   (a) Let X be the Hilbert space $L^2(0,\infty)$ with the usual inner product. Define the operator A by

$$(Af)(x) = x^2 f(x), \quad \mathcal{D}(A) = \{f \in X \mid \int_0^\infty |x^2 f(x)|^2 \, dx < \infty\}.$$

   (b) Let X be a Hilbert space, and let A be an orthogonal projection on X. (Here's are reminder of the definition of an orthogonal projection: Let M be a closed subspace X. Every $z \in M$ can be written uniquely as $z = x + y$ with $x \in M$ and $y \in M^\perp$. Then $Az = x$ is an orthogonal projection.)

3. Show that any weakly convergent sequence in a Banach space is a bounded sequence.

4. Let $\ell^2 = \{(x_1, x_2, x_3, \ldots) \mid x_k \in \mathbf{C}$ and $\sum_{k=1}^\infty |x_k|^2 < \infty\}$, with the usual inner product. Define

$$A(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

   Find A^*. Is A a normal operator?

5. Let X be a Hilbert space, and let A be the infinitesimal generator of a strongly continuous semigroup e^{At} on X.

   (a) Suppose that $K \in \mathcal{B}(X)$ is a compact operator. Prove that

$$\lim_{t \to 0} \|(e^{At} - I)K\| = 0.$$

   (b) Show, by constructing a counterexample, that the following is *not* true for all infinitessimal generators A:

$$\lim_{t \to 0} \|(e^{At} - I)\| = 0.$$

6. With $\Omega \subset \mathbb{R}^n$ a bounded open set, consider the boundary value problem

$$\begin{aligned} -\Delta u &= f \text{ on } \Omega \\ \frac{\partial u}{\partial \nu} &= g \text{ on } \Gamma, \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

   (a) What are the appropriate choices of a Hilbert space V, a bilinear form $a(\cdot,\cdot)$, and a linear functional $F(\cdot)$ for the variational formulation of this problem?

   (b) Show that a solution exists only if

$$\int_\Omega f d\Omega = -\int_\Gamma g d\Gamma.$$

   Also show that if u is a solution of (1) then so is $u + c$, where constant c is arbitrary.

7. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the equation

$$-\Delta u = u^2,$$

where Ω is a bounded, connected open set in $\mathbb{R}^n$, use the Strong Maximum Principle to prove that u cannot attain its infimum on Ω unless u is identically zero.

8. Recalling the Lax-Milgram Theorem, let

$$\begin{aligned} V &= H_0^1(0,1), \\ a(u,v) &= \int_0^1 [u'v' + c(x)u'v]\, dx \text{ for } u, v \in V \\ F(v) &= \int_0^1 fv dx \text{ for } v \in V, \end{aligned}$$

where $f \in L^2(0,1)$, and $|c(x)| \leq c_1$, $0 \leq x \leq 1$.

   (a) Show that $a(\cdot,\cdot)$ is a bounded, unsymmetric bilinear form.

   (b) Show that $a(\cdot,\cdot)$ is V-elliptic if c_1 is sufficiently small, or if $c(x)$ is a constant function.

   (c) Find some $c(x)$ and $u \in H_0^1(0,1)$ such $a(u,u) < 0$.

   (d) What is the boundary value problem associated with the variational problem, "find $u \in V$ such that u solves

$$a(u,v) = F(v) \text{ for all } v \in V\text{"?}$$